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Fractions

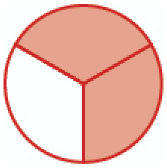
By the end of this section, you will be able to:

- Simplify fractions
- Multiply and divide fractions
- Add and subtract fractions
- Use the order of operations to simplify fractions
- Evaluate variable expressions with fractions

Note:
A more thorough introduction to the topics covered in this section can be found in the *Elementary Algebra* chapter, Foundations.

Simplify Fractions

A **fraction** is a way to represent parts of a whole. The fraction $\frac{2}{3}$ represents two of three equal parts. See [link]. In the fraction $\frac{2}{3}$, the 2 is called the **numerator** and the 3 is called the **denominator**. The line is called the fraction bar.



In the circle, $\frac{2}{3}$ of the circle is shaded—2 of the 3 equal parts.

Note:
Fraction
A **fraction** is written $\frac{a}{b}$, where $b \neq 0$ and
a is the **numerator** and b is the **denominator**.

A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

Fractions that have the same value are **equivalent fractions**. The Equivalent Fractions

Property allows us to find equivalent fractions and also simplify fractions.

Note:
Equivalent Fractions Property
If a, b, and c are numbers where $b \neq 0, c \neq 0$,
then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator.

For example,

$\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.

$\frac{10}{15}$ is not simplified because 5 is a common factor of 10 and 15.

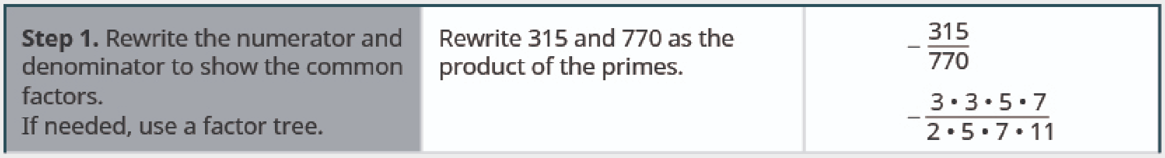
We simplify, or reduce, a fraction by removing the common factors of the numerator and denominator. A fraction is not simplified until all common factors have been removed. If an expression has fractions, it is not completely simplified until the fractions are simplified.

Sometimes it may not be easy to find common factors of the numerator and denominator. When this happens, a good idea is to factor the numerator and the denominator into prime numbers. Then divide out the common factors using the Equivalent Fractions Property.

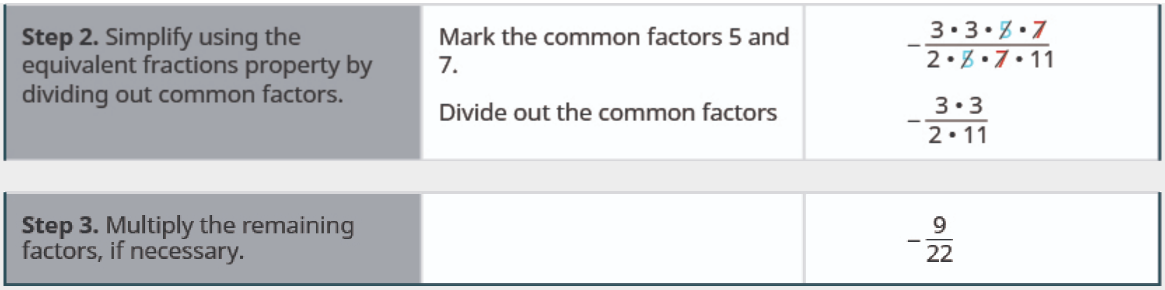
Example:
How To Simplify a Fraction
Exercise:

Problem: Simplify: $-\frac{315}{770}$.

Solution:



Step 1. Rewrite the numerator and denominator to show the common factors. If needed, use a factor tree.	Rewrite 315 and 770 as the product of the primes.	$-\frac{315}{770}$ $-\frac{3 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 5 \cdot 7 \cdot 11}$



Step 2. Simplify using the equivalent fractions property by dividing out common factors.	Mark the common factors 5 and 7. Divide out the common factors	$-\frac{3\cdot 3\cdot \cancel{5}\cdot \cancel{7}}{2\cdot \cancel{5}\cdot \cancel{7}\cdot 11}$ $-\frac{3\cdot 3}{2\cdot 11}$
Step 3. Multiply the remaining factors, if necessary.		$-\frac{9}{22}$

Note:
Exercise:

Problem: Simplify: $-\frac{69}{120}$.

Solution:

$-\frac{23}{40}$

Note:
Exercise:

Problem: Simplify: $-\frac{120}{192}$.

Solution:

$-\frac{5}{8}$

We now summarize the steps you should follow to simplify fractions.

Note:
Simplify a fraction.

Rewrite the numerator and denominator to show the common factors.	If needed, factor the numerator and denominator into prime numbers first.
Simplify using the Equivalent Fractions Property by dividing out common factors.	
Multiply any remaining factors.	

Multiply and Divide Fractions

Many people find multiplying and dividing fractions easier than adding and subtracting fractions.

To multiply fractions, we multiply the numerators and multiply the denominators.

Note:
Fraction Multiplication
If a, b, c, and d are numbers where $b \neq 0$, and $d \neq 0$, then
Equation:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

To multiply fractions, multiply the numerators and multiply the denominators.

When multiplying fractions, the properties of positive and negative numbers still apply, of course. It is a good idea to determine the sign of the product as the first step. In [link], we will multiply negative and a positive, so the product will be negative.

When multiplying a fraction by an integer, it may be helpful to write the integer as a fraction. Any integer, a, can be written as $\frac{a}{1}$. So, for example, $3 = \frac{3}{1}$.

Example:
Exercise:

Problem: Multiply: $-\frac{12}{5}(-20x)$.

Solution:

The first step is to find the sign of the product. Since the signs are the same, the product is positive.

	$-\frac{12}{5}(-20x)$
Determine the sign of the product. The signs are the same, so the product is positive.	$\frac{12}{5}(20x)$

Write 20*x* as a fraction.	$\frac{12}{5}\left(\frac{20x}{1}\right)$
Multiply.	$\frac{12\cdot 20x}{5\cdot 1}$
Rewrite 20 to show the common factor 5 and divide it out.	$\frac{12\cdot 4\cdot \cancel{5}\cdot x}{\cancel{5}\cdot 1}$
Simplify.	$48x$

Note:
Exercise:

Problem: Multiply: $\frac{11}{3}(-9a)$.

Solution:

$-33a$

Note:
Exercise:

Problem: Multiply: $\frac{13}{7}(-14b)$.

Solution:

$-26b$

Now that we know how to multiply fractions, we are almost ready to divide. Before we can do that, we need some vocabulary. The **reciprocal** of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. Since 4 is written in fraction form as $\frac{4}{1}$, the reciprocal of 4 is $\frac{1}{4}$.

To divide fractions, we multiply the first fraction by the reciprocal of the second.

Note:
Fraction Division
If a, b, c, and d are numbers where $b \neq 0$, $c \neq 0$, and $d \neq 0$, then
Equation:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

To divide fractions, we multiply the first fraction by the **reciprocal** of the second.

We need to say $b \neq 0$, $c \neq 0$, and $d \neq 0$, to be sure we don’t divide by zero!

Example:
Exercise:

Problem: Find the quotient: $-\frac{7}{18} \div \left(-\frac{14}{27}\right)$.

Solution:

	$-\frac{7}{18} \div \left(-\frac{14}{27}\right)$
To divide, multiply the first fraction by the reciprocal of the second.	$-\frac{7}{18}\left(-\frac{27}{14}\right)$
Determine the sign of the product, and then multiply.	$\frac{7 \cdot 27}{18 \cdot 14}$
Rewrite showing common factors.	$\frac{\cancel{7} \cdot \cancel{9} \cdot 3}{\cancel{9} \cdot 2 \cdot \cancel{7} \cdot 2}$
Remove common factors.	

	$\frac{3}{2 \cdot 2}$
Simplify.	$\frac{3}{4}$

Note:
Exercise:

Problem: Divide: $-\frac{7}{27} \div \left(-\frac{35}{36}\right)$.

Solution:

$\frac{4}{15}$

Note:
Exercise:

Problem: Divide: $-\frac{5}{14} \div \left(-\frac{15}{28}\right)$.

Solution:

$\frac{2}{3}$

The numerators or denominators of some fractions contain fractions themselves. A fraction in which the numerator or the denominator is a fraction is called a **complex fraction**.

Note:
Complex Fraction
A **complex fraction** is a fraction in which the numerator or the denominator contains a fraction.

Some examples of complex fractions are:
Equation:

$$\frac{\frac{6}{7}}{3} \qquad \frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{5}{6}}$$

To simplify a complex fraction, remember that the fraction bar means division. For example, the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ means $\frac{3}{4} \div \frac{5}{8}$.

Example:
Exercise:

Problem: Simplify: $\frac{\frac{x}{2}}{\frac{xy}{6}}$.

Solution:

	$\frac{\frac{x}{2}}{\frac{xy}{6}}$
Rewrite as division.	$\frac{x}{2} \div \frac{xy}{6}$
Multiply the first fraction by the reciprocal of the second.	$\frac{x}{2} \cdot \frac{6}{xy}$
Multiply.	$\frac{x \cdot 6}{2 \cdot xy}$
Look for common factors.	$\frac{\cancel{x} \cdot 3 \cdot \cancel{2}}{\cancel{2} \cdot \cancel{x} \cdot y}$
Divide common factors and simplify.	$\frac{3}{y}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{a}{8}}{\frac{ab}{6}}$.

Solution:

$\frac{3}{4b}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{p}{2}}{\frac{pq}{8}}$.

Solution:

$\frac{4}{q}$

Add and Subtract Fractions

When we multiplied fractions, we just multiplied the numerators and multiplied the denominators right straight across. To add or subtract fractions, they must have a common denominator.

Note:
Fraction Addition and Subtraction
If a, b, and c are numbers where $c \neq 0$, then
Equation:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \text{ and } \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

The **least common denominator** (LCD) of two fractions is the smallest number that can be used as a common denominator of the fractions. The LCD of the two fractions is the least common multiple (LCM) of their denominators.

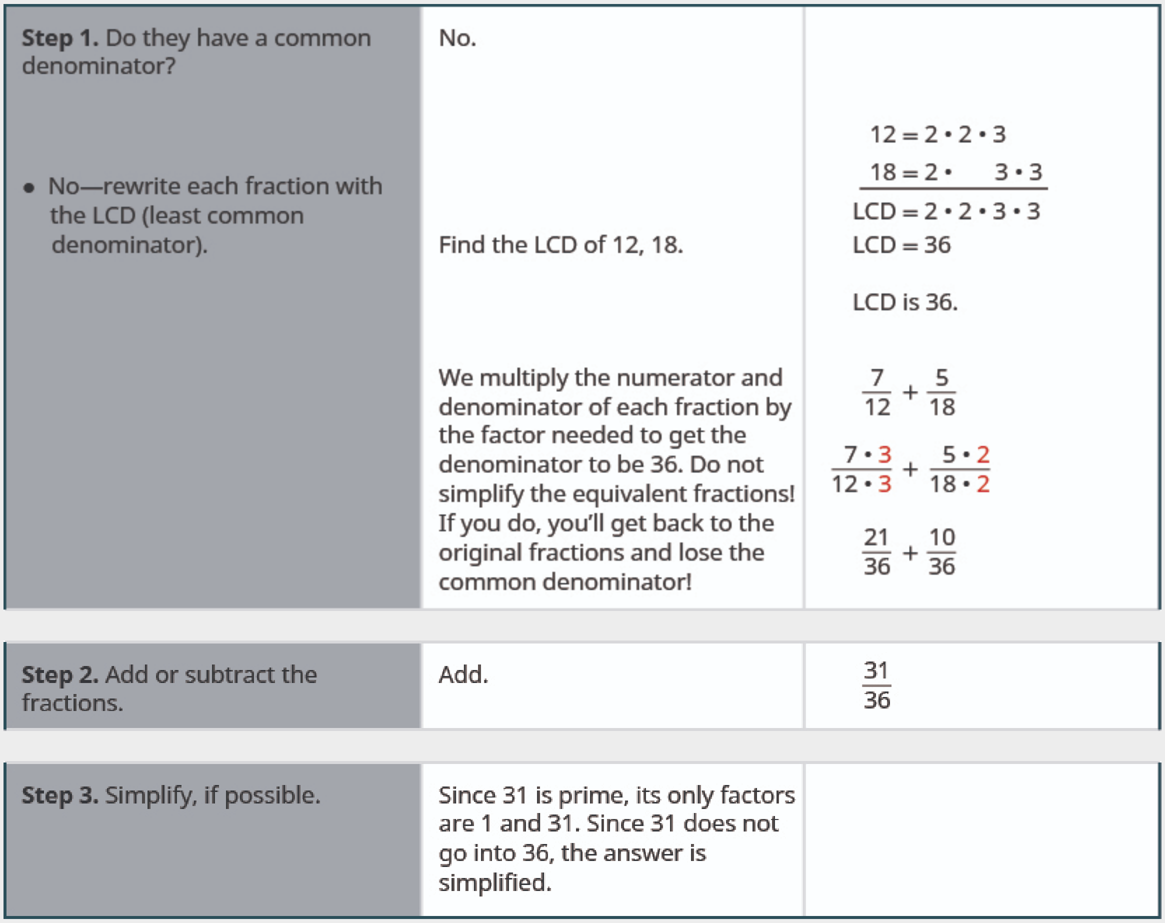
Note:
Least Common Denominator
The **least common denominator** (LCD) of two fractions is the least common multiple (LCM) of their denominators.

After we find the least common denominator of two fractions, we convert the fractions to equivalent fractions with the LCD. Putting these steps together allows us to add and subtract fractions because their denominators will be the same!

Example:
How to Add or Subtract Fractions
Exercise:

Problem: Add: $\frac{7}{12} + \frac{5}{18}$.

Solution:



Step 1. Do they have a common denominator?	No.	
- No—rewrite each fraction with the LCD (least common denominator).	Find the LCD of 12, 18.	$12 = 2\cdot 2\cdot 3$ $\underline{18 = 2\cdot \quad 3\cdot 3}$ $\text{LCD} = 2\cdot 2\cdot 3\cdot 3$ $\text{LCD} = 36$ LCD is 36.
	We multiply the numerator and denominator of each fraction by the factor needed to get the denominator to be 36. Do not simplify the equivalent fractions! If you do, you'll get back to the original fractions and lose the common denominator!	$\frac{7}{12} + \frac{5}{18}$ $\frac{7\cdot 3}{12\cdot 3} + \frac{5\cdot 2}{18\cdot 2}$ $\frac{21}{36} + \frac{10}{36}$
Step 2. Add or subtract the fractions.	Add.	$\frac{31}{36}$
Step 3. Simplify, if possible.	Since 31 is prime, its only factors are 1 and 31. Since 31 does not go into 36, the answer is simplified.	

Note:
Exercise:

Problem: Add: $\frac{7}{12} + \frac{11}{15}$.

Solution:

$\frac{79}{60}$

Note:
Exercise:

Problem: Add: $\frac{13}{15} + \frac{17}{20}$.

Solution:

$\frac{103}{60}$

Note:
Add or subtract fractions.

Do they have a common denominator?

- Yes—go to step 2.
- No—rewrite each fraction with the LCD (least common denominator).
    - Find the LCD.
    - Change each fraction into an equivalent fraction with the LCD as its denominator.

Add or subtract the fractions.
Simplify, if possible.

We now have all four operations for fractions. [link] summarizes fraction operations.

Fraction Multiplication	**Fraction Division**
$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$
Multiply the numerators and multiply the denominators	Multiply the first fraction by the reciprocal of the second.
Fraction Addition	**Fraction Subtraction**
$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$	$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$
Add the numerators and place the sum over the common denominator.	Subtract the numerators and place the difference over the common denominator.
To multiply or divide fractions, an LCD is NOT needed. To add or subtract fractions, an LCD is needed.	

When starting an exercise, always identify the operation and then recall the methods needed for that operation.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{5x}{6} - \frac{3}{10}$ ⓑ $\frac{5x}{6} \cdot \frac{3}{10}$.

Solution:

First ask, “What is the operation?” Identifying the operation will determine whether or not we need a common denominator. Remember, we need a common denominator to add or subtract, but not to multiply or divide.

ⓐ

What is the operation? The operation is subtraction.

Do the fractions have a common denominator? No. $\frac{5x}{6} - \frac{3}{10}$

Find the LCD of 6 and 10 The LCD is 30.

$$\begin{aligned} 6 &= 2 \cdot 3 \\ \underline{10} &\underline{= 2 \cdot 5} \\ \text{LCD} &= 2 \cdot 3 \cdot 5 \\ \text{LCD} &= 30 \end{aligned}$$

Rewrite each fraction as an equivalent fraction with the LCD. $\frac{5x \cdot 5}{6 \cdot 5} - \frac{3 \cdot 3}{10 \cdot 3}$

$$\frac{25x}{30} - \frac{9}{30}$$

Subtract the numerators and place the difference over the common denominators. $\frac{25x - 9}{30}$

Simplify, if possible. There are no common factors. The fraction is simplified.

ⓑ

What is the operation? Multiplication. $\frac{25x}{6} \cdot \frac{3}{10}$

To multiply fractions, multiply the numerators and multiply the denominators. $\frac{25x \cdot 3}{6 \cdot 10}$

Rewrite, showing common factors.

Remove common factors. $\frac{\cancel{5}\, x \cdot \cancel{3}}{2 \cdot \cancel{3} \cdot 2 \cdot \cancel{5}}$

Simplify. $\frac{x}{4}$

Notice, we needed an LCD to add $\frac{25x}{6} - \frac{3}{10}$, but not to multiply $\frac{25x}{6} \cdot \frac{3}{10}$.

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{3a}{4} - \frac{8}{9}$ ⓑ $\frac{3a}{4} \cdot \frac{8}{9}$.

Solution:

ⓐ $\frac{27a-32}{36}$ ⓑ $\frac{2a}{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{4k}{5} - \frac{1}{6}$ ⓑ $\frac{4k}{5} \cdot \frac{1}{6}$.

Solution:

ⓐ $\frac{24k-5}{30}$ ⓑ $\frac{2k}{15}$

Use the Order of Operations to Simplify Fractions

The fraction bar in a fraction acts as grouping symbol. The order of operations then tells us to simplify the numerator and then the denominator. Then we divide.

Note:
Simplify an expression with a fraction bar.

Simplify the expression in the numerator. Simplify the expression in the denominator.
Simplify the fraction.

Where does the negative sign go in a fraction? Usually the negative sign is in front of the fraction, but you will sometimes see a fraction with a negative numerator, or sometimes with a negative denominator. Remember that fractions represent division. When the numerator and denominator have different signs, the quotient is negative.
Equation:

$$\frac{-1}{3} = -\frac{1}{3} \qquad \frac{\text{negative}}{\text{positive}} = \text{negative}$$

Equation:

$$\frac{1}{-3} = -\frac{1}{3} \qquad \frac{\text{positive}}{\text{negative}} = \text{negative}$$

Note:
Placement of Negative Sign in a Fraction
For any positive numbers *a* and *b*,
Equation:

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

Example:
Exercise:

Problem: Simplify: $\frac{4(-3)+6(-2)}{-3(2)-2}$.

Solution:

The fraction bar acts like a grouping symbol. So completely simplify the numerator and the denominator separately.

	$\frac{4(-3)+6(-2)}{-3(2)-2}$
Multiply.	$\frac{-12+(-12)}{-6-2}$
Simplify.	$\frac{-24}{-8}$
Divide.	3

Note:
Exercise:

Problem: Simplify: $\frac{8(-2)+4(-3)}{-5(2)+3}$.

Solution:

4

Note:

Exercise:

Problem: Simplify: $\frac{7(-1)+9(-3)}{-5(3)-2}$.

Solution:

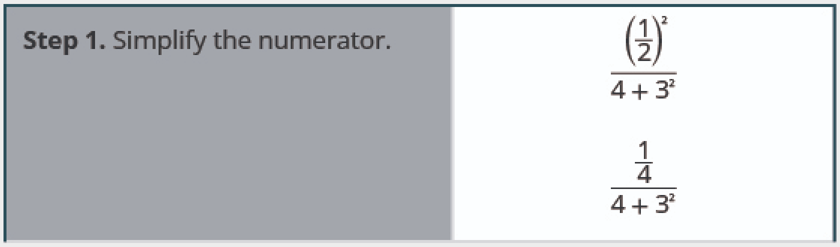
2

Now we’ll look at complex fractions where the numerator or denominator contains an expression that can be simplified. So we first must completely simplify the numerator and denominator separately using the order of operations. Then we divide the numerator by the denominator as the fraction bar means division.

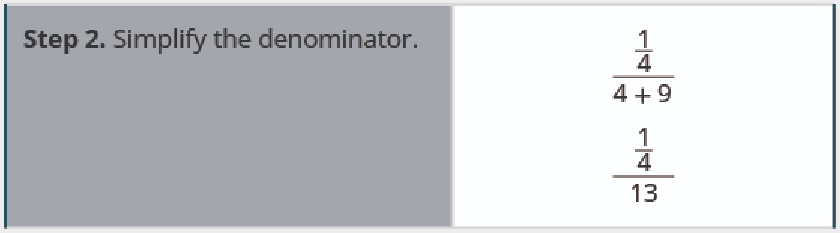
Example:
How to Simplify Complex Fractions
Exercise:

Problem: Simplify: $\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$.

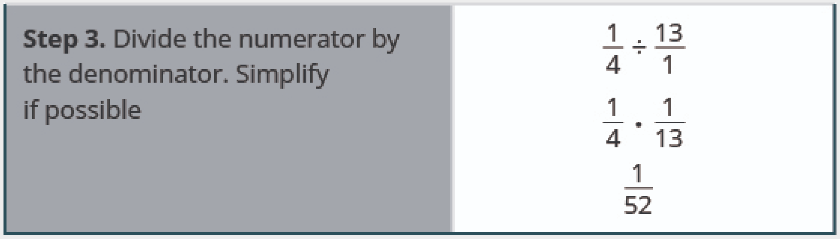
Solution:



Step 1. Simplify the numerator.	$\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$ $\frac{\frac{1}{4}}{4+3^2}$



Step 2. Simplify the denominator.	$\frac{\frac{1}{4}}{4+9}$ $\frac{\frac{1}{4}}{13}$



Step 3. Divide the numerator by the denominator. Simplify if possible	$\frac{1}{4} \div \frac{13}{1}$ $\frac{1}{4} \cdot \frac{1}{13}$ $\frac{1}{52}$

Note:
Exercise:

Problem: Simplify: $\frac{\left(\frac{1}{3}\right)^2}{2^3+2}$.

Solution:

$\frac{1}{90}$

Note:
Exercise:

Problem: Simplify: $\frac{1+4^2}{\left(\frac{1}{4}\right)^2}$.

Solution:

272

Note:
Simplify complex fractions.

Simplify the numerator.
Simplify the denominator.
Divide the numerator by the denominator. Simplify if possible.

Example:
Exercise:

Problem: Simplify: $\frac{\frac{1}{2}+\frac{2}{3}}{\frac{3}{4}-\frac{1}{6}}$.

Solution:

It may help to put parentheses around the numerator and the denominator.

	$\frac{\left(\frac{1}{2}+\frac{2}{3}\right)}{\left(\frac{3}{4}-\frac{1}{6}\right)}$
Simplify the numerator $(\text{LCD} = 6)$ and simplify the denominator $(\text{LCD} = 12)$.	$\frac{\left(\frac{3}{6}+\frac{4}{6}\right)}{\left(\frac{9}{12}-\frac{2}{12}\right)}$
Simplify.	$\frac{\left(\frac{7}{6}\right)}{\left(\frac{7}{12}\right)}$
Divide the numerator by the denominator.	$\frac{7}{6} \div \frac{7}{12}$
Simplify.	$\frac{7}{6} \cdot \frac{12}{7}$
Divide out common factors.	$\frac{\cancel{7}\cdot\cancel{6}\cdot 2}{\cancel{6}\cdot\cancel{7}\cdot 1}$
Simplify.	2

Note:
Exercise:

Problem: Simplify: $\frac{\frac{1}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{1}{3}}$.

Solution:

2

Note:
Exercise:

Problem: Simplify: $\frac{\frac{2}{3}-\frac{1}{2}}{\frac{1}{4}+\frac{1}{3}}$.

Solution:

$\frac{2}{7}$

Evaluate Variable Expressions with Fractions

We have evaluated expressions before, but now we can evaluate expressions with fractions. Remember, to evaluate an expression, we substitute the value of the variable into the expression and then simplify.

Example:

Exercise:

Problem: Evaluate $2x^2y$ when $x = \frac{1}{4}$ and $y = -\frac{2}{3}$.

Solution:

Substitute the values into the expression.

	$2x^2y$
Substitute $\frac{1}{4}$ for *x* and $-\frac{2}{3}$ for *y*.	$2\left(\frac{1}{4}\right)^2\left(-\frac{2}{3}\right)$
Simplify exponents first.	$2\left(\frac{1}{16}\right)\left(-\frac{2}{3}\right)$
Multiply; divide out the common factors. Notice we write 16 as $2 \cdot 2 \cdot 4$ to make it easy to remove common factors.	$-\frac{\cancel{2}\cdot 1\cdot \cancel{2}}{\cancel{2}\cdot \cancel{2}\cdot 4\cdot 3}$
Simplify.	$-\frac{1}{12}$

Note:
Exercise:

Problem: Evaluate $3ab^2$ when $a = -\frac{2}{3}$ and $b = -\frac{1}{2}$.

Solution:

$-\frac{1}{2}$

Note:
Exercise:

Problem: Evaluate $4c^3d$ when $c = -\frac{1}{2}$ and $d = -\frac{4}{3}$.

Solution:

$\frac{2}{3}$

Note:
Access this online resource for additional instruction and practice with fractions.

- Adding Fractions with Unlike Denominators

Key Concepts

- **Equivalent Fractions Property**
  If a, b, and c are numbers where $b \neq 0, c \neq 0$, then
  $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.
- **How to simplify a fraction.**

  Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers first.
  Simplify using the Equivalent Fractions Property by dividing out common factors.
  Multiply any remaining factors.

- **Fraction Multiplication**
  If a, b, c, and d are numbers where $b \neq 0$, and $d \neq 0$, then
  $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.
  To multiply fractions, multiply the numerators and multiply the denominators.
- **Fraction Division**
  If a, b, c, and d are numbers where $b \neq 0, c \neq 0$, and $d \neq 0$, then
  $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.
  To divide fractions, we multiply the first fraction by the reciprocal of the second.
- **Fraction Addition and Subtraction**
  If a, b, and c are numbers where $c \neq 0$, then
  $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ and $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$.
  To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.
- **How to add or subtract fractions.**

  Do they have a common denominator?
    - Yes—go to step 2.

        - No—rewrite each fraction with the LCD (least common denominator).
            - Find the LCD.
            - Change each fraction into an equivalent fraction with the LCD as its denominator.

    Add or subtract the fractions.
    Simplify, if possible.

- **How to simplify an expression with a fraction bar.**

  Simplify the expression in the numerator. Simplify the expression in the denominator.
  Simplify the fraction.

- **Placement of Negative Sign in a Fraction**
  For any positive numbers *a* and *b*,
  $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$.
- **How to simplify complex fractions.**

  Simplify the numerator.
  Simplify the denominator.
  Divide the numerator by the denominator. Simplify if possible.

Practice Makes Perfect

Simplify Fractions

In the following exercises, simplify.
Exercise:

Problem: $-\frac{108}{63}$

Solution:

$-\frac{12}{7}$

Exercise:

Problem: $-\frac{104}{48}$

Exercise:

Problem: $\frac{120}{252}$

Solution:

$\frac{10}{21}$

Exercise:

Problem: $\frac{182}{294}$

Exercise:

Problem: $\frac{14x^2}{21y}$

Solution:

$\frac{2x^2}{3y}$

Exercise:

Problem: $\frac{24a}{32b^2}$

Exercise:

Problem: $-\frac{210a^2}{110b^2}$

Solution:

$-\frac{21a^2}{11b^2}$

Exercise:

Problem: $-\frac{30x^2}{105y^2}$

Multiply and Divide Fractions

In the following exercises, perform the indicated operation.

Exercise:

Problem: $-\frac{3}{4}\left(-\frac{4}{9}\right)$

Solution:

$\frac{1}{3}$

Exercise:

Problem: $-\frac{3}{8} \cdot \frac{4}{15}$

Exercise:

Problem: $\left(-\frac{14}{15}\right)\left(\frac{9}{20}\right)$

Solution:

$-\frac{21}{50}$

Exercise:

Problem: $\left(-\frac{9}{10}\right)\left(\frac{25}{33}\right)$

Exercise:

Problem: $\left(-\frac{63}{84}\right)\left(-\frac{44}{90}\right)$

Solution:

$\frac{11}{30}$

Exercise:

Problem: $\left(-\frac{33}{60}\right)\left(-\frac{40}{88}\right)$

Exercise:

Problem: $\frac{3}{7}\cdot 21n$

Solution:

$9n$

Exercise:

Problem: $\frac{5}{6}\cdot 30m$

Exercise:

Problem: $\frac{3}{4}\div\frac{x}{11}$

Solution:

$\frac{33}{4x}$

Exercise:

Problem: $\frac{2}{5}\div\frac{y}{9}$

Exercise:

Problem: $\frac{5}{18}\div\left(-\frac{15}{24}\right)$

Solution:

$-\frac{4}{9}$

Exercise:

Problem: $\frac{7}{18} \div \left(-\frac{14}{27}\right)$

Exercise:

Problem: $\frac{8u}{15} \div \frac{12v}{25}$

Solution:

$\frac{10u}{9v}$

Exercise:

Problem: $\frac{12r}{25} \div \frac{18s}{35}$

Exercise:

Problem: $\frac{3}{4} \div (-12)$

Solution:

$-\frac{1}{16}$

Exercise:

Problem: $-15 \div \left(-\frac{5}{3}\right)$

In the following exercises, simplify.

Exercise:

Problem: $\frac{-\frac{8}{21}}{\frac{12}{35}}$

Solution:

$-\frac{10}{9}$

Exercise:

Problem: $\frac{-\frac{9}{16}}{\frac{33}{40}}$

Exercise:

Problem: $\frac{-\frac{4}{5}}{2}$

Solution:

$-\frac{2}{5}$

Exercise:

Problem: $\frac{\frac{5}{3}}{10}$

Exercise:

Problem: $\frac{\frac{m}{3}}{\frac{n}{2}}$

Solution:

$\frac{2m}{3n}$

Exercise:

Problem: $\frac{-\frac{3}{8}}{-\frac{y}{12}}$

Add and Subtract Fractions

In the following exercises, add or subtract.

Exercise:

Problem: $\frac{7}{12}+\frac{5}{8}$

Solution:

$\frac{29}{24}$

Exercise:

Problem: $\frac{5}{12}+\frac{3}{8}$

Exercise:

Problem: $\frac{7}{12}-\frac{9}{16}$

Solution:

$\frac{1}{48}$

Exercise:

Problem: $\frac{7}{16}-\frac{5}{12}$

Exercise:

Problem: $-\frac{13}{30}+\frac{25}{42}$

Solution:

$\frac{17}{105}$

Exercise:

Problem: $-\frac{23}{30}+\frac{5}{48}$

Exercise:

Problem: $-\frac{39}{56}-\frac{22}{35}$

Solution:

$-\frac{53}{40}$

Exercise:

Problem: $-\frac{33}{49}-\frac{18}{35}$

Exercise:

Problem: $-\frac{2}{3}-\left(-\frac{3}{4}\right)$

Solution:

$\frac{1}{12}$

Exercise:

Problem: $-\frac{3}{4}-\left(-\frac{4}{5}\right)$

Exercise:

Problem: $\frac{x}{3}+\frac{1}{4}$

Solution:

$\frac{4x+3}{12}$

Exercise:

Problem: $\frac{x}{5}-\frac{1}{4}$

Exercise:

ⓐ $\frac{2}{3}+\frac{1}{6}$

Problem: ⓑ $\frac{2}{3}\div\frac{1}{6}$

Solution:

ⓐ $\frac{5}{6}$ ⓑ 4

Exercise:

ⓐ $-\frac{2}{5}-\frac{1}{8}$

Problem: ⓑ $-\frac{2}{5}\cdot\frac{1}{8}$

Exercise:

ⓐ $\frac{5n}{6}\div\frac{8}{15}$

Problem: ⓑ $\frac{5n}{6}-\frac{8}{15}$

Solution:

ⓐ $\frac{25n}{16}$ ⓑ $\frac{25n-16}{30}$

Exercise:

ⓐ $\frac{3a}{8}\div\frac{7}{12}$

Problem: ⓑ $\frac{3a}{8}-\frac{7}{12}$

Exercise:

ⓐ $-\frac{4x}{9}-\frac{5}{6}$

Problem: ⓑ $-\frac{4k}{9}\cdot\frac{5}{6}$

Solution:

ⓐ $\frac{-8x-15}{18}$ ⓑ $-\frac{10k}{27}$

Exercise:

ⓐ $-\frac{3y}{8}-\frac{4}{3}$

Problem: ⓑ $-\frac{3y}{8}\cdot\frac{4}{3}$

Exercise:

Problem: ⓐ $-\frac{5a}{3}+\left(-\frac{10}{6}\right)$
ⓑ $-\frac{5a}{3}\div\left(-\frac{10}{6}\right)$

Solution:

ⓐ $\frac{-5(a+1)}{3}$ ⓑ a

Exercise:

Problem: ⓐ $\frac{2b}{5}+\frac{8}{15}$
ⓑ $\frac{2b}{5}\div\frac{8}{15}$

Use the Order of Operations to Simplify Fractions

In the following exercises, simplify.

Exercise:

Problem: $\frac{5\cdot 6-3\cdot 4}{4\cdot 5-2\cdot 3}$

Solution:

$\frac{9}{7}$

Exercise:

Problem: $\frac{8\cdot 9-7\cdot 6}{5\cdot 6-9\cdot 2}$

Exercise:

Problem: $\frac{5^2-3^2}{3-5}$

Solution:

-8

Exercise:

Problem: $\frac{6^2-4^2}{4-6}$

Exercise:

Problem: $\frac{7\cdot 4-2(8-5)}{9\cdot 3-3\cdot 5}$

Solution:

$\frac{11}{6}$

Exercise:

Problem: $\frac{9 \cdot 7 - 3(12 - 8)}{8 \cdot 7 - 6 \cdot 6}$

Exercise:

Problem: $\frac{9(8-2)-3(15-7)}{6(7-1)-3(17-9)}$

Solution:

$\frac{5}{2}$

Exercise:

Problem: $\frac{8(9-2)-4(14-9)}{7(8-3)-3(16-9)}$

Exercise:

Problem: $\frac{2^3+4^2}{\left(\frac{2}{3}\right)^2}$

Solution:

54

Exercise:

Problem: $\frac{3^3-3^2}{\left(\frac{3}{4}\right)^2}$

Exercise:

Problem: $\frac{\left(\frac{3}{5}\right)^2}{\left(\frac{3}{7}\right)^2}$

Solution:

$\frac{49}{25}$

Exercise:

Problem: $\frac{\left(\frac{3}{4}\right)^2}{\left(\frac{5}{8}\right)^2}$

Exercise:

Problem: $\frac{2}{\frac{1}{3}+\frac{1}{5}}$

Solution:

$\frac{15}{4}$

Exercise:

Problem: $\frac{5}{\frac{1}{4}+\frac{1}{3}}$

Exercise:

Problem: $\frac{\frac{7}{8}-\frac{2}{3}}{\frac{1}{2}+\frac{3}{8}}$

Solution:

$\frac{5}{21}$

Exercise:

Problem: $\frac{\frac{3}{4}-\frac{3}{5}}{\frac{1}{4}+\frac{2}{5}}$

Mixed Practice

In the following exercises, simplify.

Exercise:

Problem: $-\frac{3}{8} \div \left(-\frac{3}{10}\right)$

Solution:

$\frac{5}{4}$

Exercise:

Problem: $-\frac{3}{12} \div \left(-\frac{5}{9}\right)$

Exercise:

Problem: $-\frac{3}{8} + \frac{5}{12}$

Solution:

$\frac{1}{24}$

Exercise:

Problem: $-\frac{1}{8} + \frac{7}{12}$

Exercise:

Problem: $-\frac{7}{15} - \frac{y}{4}$

Solution:

$\frac{-28-15y}{60}$

Exercise:

Problem: $-\frac{3}{8} - \frac{x}{11}$

Exercise:

Problem: $\frac{11}{12a} \cdot \frac{9a}{16}$

Solution:

$\frac{33}{64}$

Exercise:

Problem: $\frac{10y}{13} \cdot \frac{8}{15y}$

Exercise:

Problem: $\frac{1}{2} + \frac{2}{3} \cdot \frac{5}{12}$

Solution:

$\frac{7}{9}$

Exercise:

Problem: $\frac{1}{3} + \frac{2}{5} \cdot \frac{3}{4}$

Exercise:

Problem: $1 - \frac{3}{5} \div \frac{1}{10}$

Solution:

-5

Exercise:

Problem: $1 - \frac{5}{6} \div \frac{1}{12}$

Exercise:

Problem: $\frac{3}{8} - \frac{1}{6} + \frac{3}{4}$

Solution:

$\frac{23}{24}$

Exercise:

Problem: $\frac{2}{5} + \frac{5}{8} - \frac{3}{4}$

Exercise:

Problem: $12\left(\frac{9}{20} - \frac{4}{15}\right)$

Solution:

$\frac{11}{5}$

Exercise:

Problem: $8\left(\frac{15}{16} - \frac{5}{6}\right)$

Exercise:

Problem: $\frac{\frac{5}{8}+\frac{1}{6}}{\frac{19}{24}}$

Solution:

1

Exercise:

Problem: $\frac{\frac{1}{6}+\frac{3}{10}}{\frac{14}{30}}$

Exercise:

Problem: $\left(\frac{5}{9} + \frac{1}{6}\right) \div \left(\frac{2}{3} - \frac{1}{2}\right)$

Solution:

$\frac{13}{3}$

Exercise:

Problem: $\left(\frac{3}{4} + \frac{1}{6}\right) \div \left(\frac{5}{8} - \frac{1}{3}\right)$

Evaluate Variable Expressions with Fractions

In the following exercises, evaluate.

Exercise:

$\frac{7}{10}-w$ when
Problem: ⓐ $w=\frac{1}{2}$ ⓑ $w=-\frac{1}{2}$

Solution:

ⓐ $\frac{1}{5}$ ⓑ $\frac{6}{5}$

Exercise:

$\frac{5}{12}-w$ when
Problem: ⓐ $w=\frac{1}{4}$ ⓑ $w=-\frac{1}{4}$

Exercise:

$2x^2y^3$ when
Problem: $x=-\frac{2}{3}$ and $y=-\frac{1}{2}$

Solution:

$-\frac{1}{9}$

Exercise:

$8u^2v^3$ when
Problem: $u=-\frac{3}{4}$ and $v=-\frac{1}{2}$

Exercise:

$\frac{a+b}{a-b}$ when
Problem: $a=-3, b=8$

Solution:

$-\frac{5}{11}$

Exercise:

$\frac{r-s}{r+s}$ when
Problem: $r=10, s=-5$

Writing Exercises

Exercise:

Problem: Why do you need a common denominator to add or subtract fractions? Explain.

Solution:

Answers will vary.

Exercise:

Problem: How do you find the LCD of 2 fractions?

Exercise:

Problem: Explain how you find the reciprocal of a fraction.

Solution:

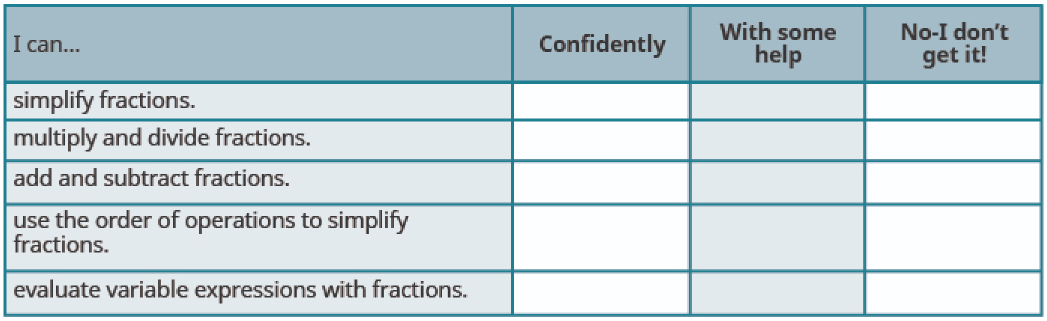
Answers will vary.

Exercise:

Problem: Explain how you find the reciprocal of a negative number.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
simplify fractions.			
multiply and divide fractions.			
add and subtract fractions.			
use the order of operations to simplify fractions.			
evaluate variable expressions with fractions.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

complex fraction

A fraction in which the numerator or the denominator is a fraction is called a complex fraction.

denominator

In a fraction, written $\frac{a}{b}$, where $b \neq 0$, the denominator b is the number of equal parts the whole has been divided into.

equivalent fractions
Equivalent fractions are fractions that have the same value.

fraction
A fraction is written $\frac{a}{b}$, where $b \neq 0$, and a is the numerator and b is the denominator. A fraction represents parts of a whole.

least common denominator
The least common denominator (LCD) of two fractions is the least common multiple (LCM) of their denominators.

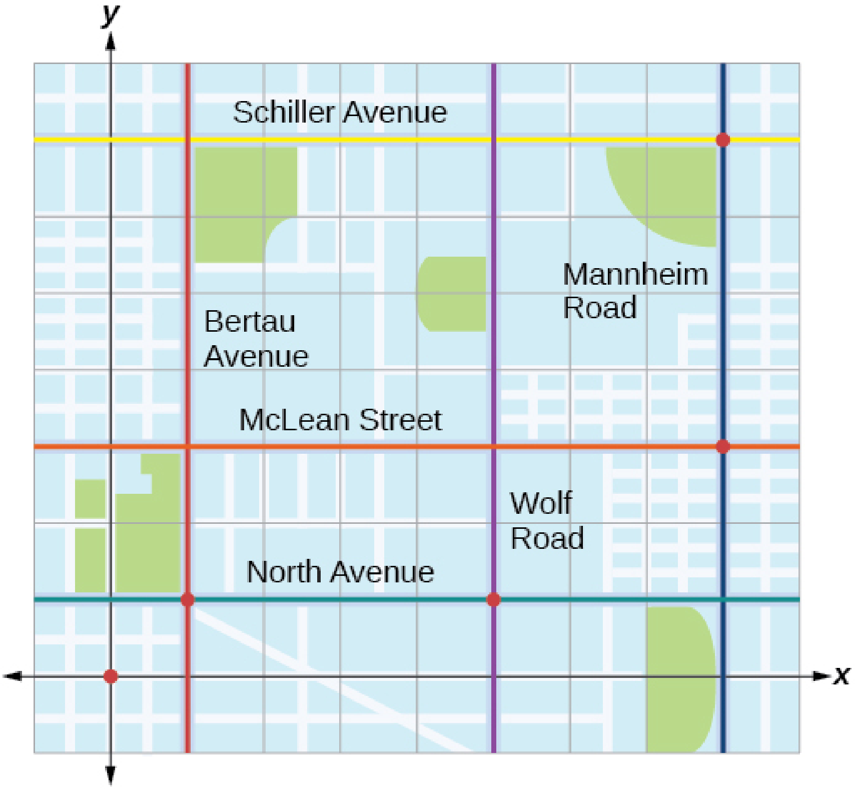
numerator
In a fraction, written $\frac{a}{b}$, where $b \neq 0$, the numerator a indicates how many parts are included.

reciprocal
The reciprocal of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator.

The Rectangular Coordinate Systems and Graphs

In this section you will:

- Plot ordered pairs in a Cartesian coordinate system.
- Graph equations by plotting points.
- Graph equations with a graphing utility.
- Find x-intercepts and y-intercepts.
- Use the distance formula.
- Use the midpoint formula.


y
x
Schiller Avenue
Mannheim Road
Bertau Avenue
McLean Street
Wolf Road
North Avenue

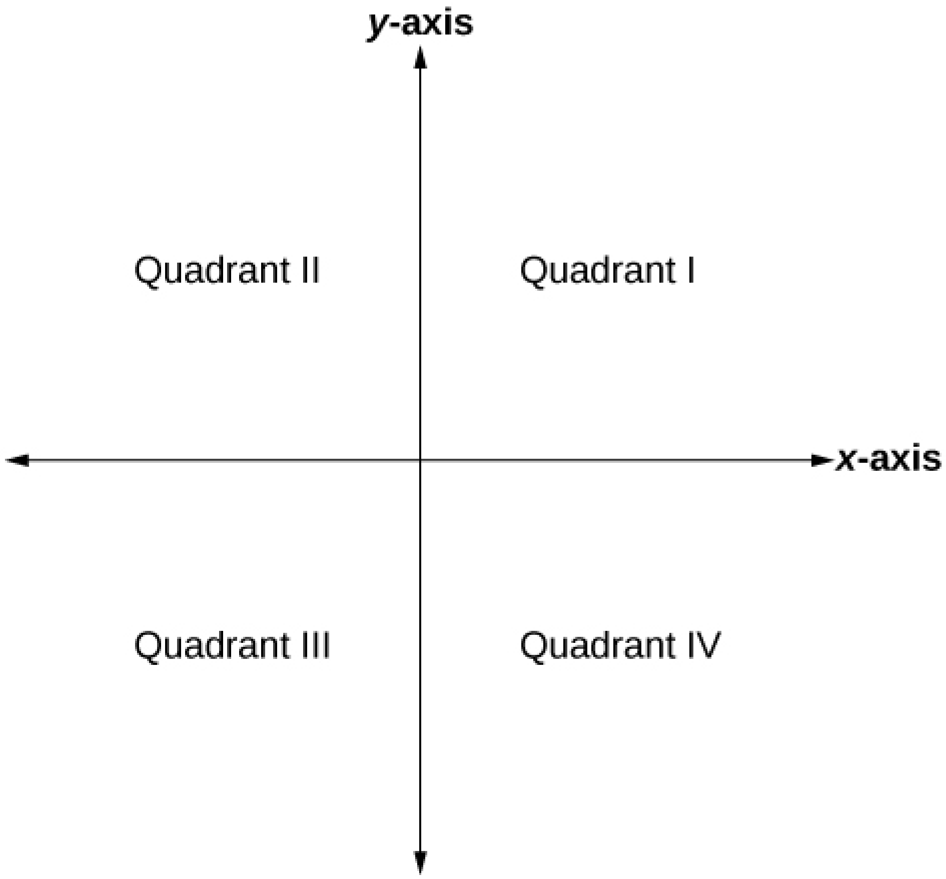
Tracie set out from Elmhurst, IL, to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot in [link]. Laying a rectangular coordinate grid over the map, we can see that each stop aligns with an intersection of grid lines. In this section, we will learn how to use grid lines to describe locations and changes in locations.

Plotting Ordered Pairs in the Cartesian Coordinate System

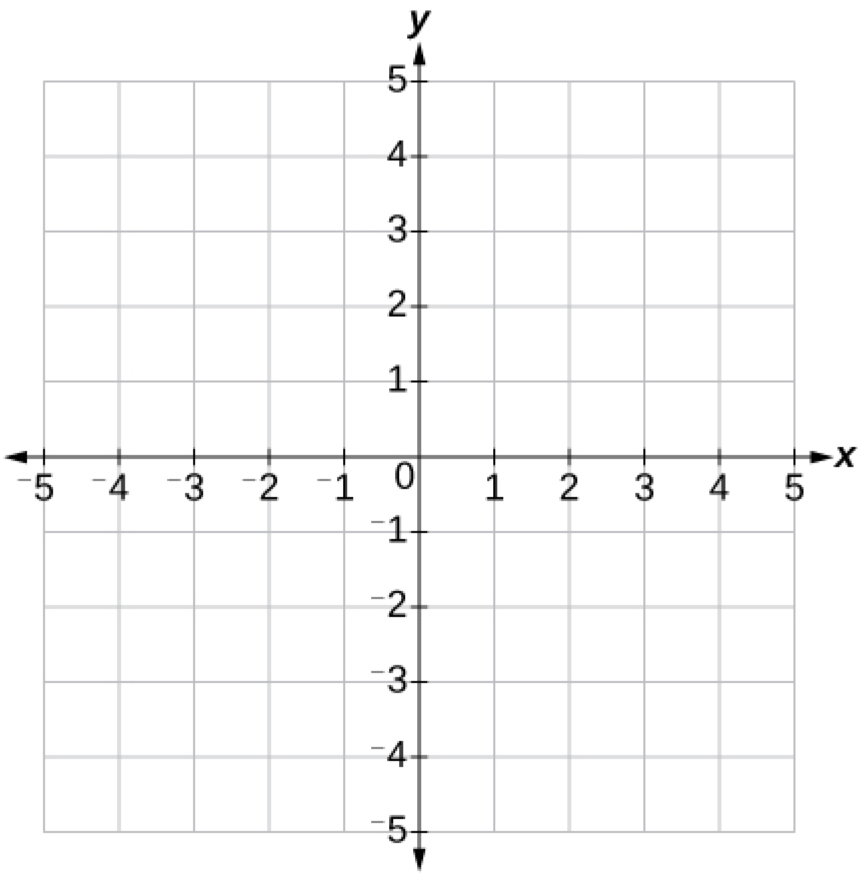
An old story describes how seventeenth-century philosopher/mathematician René Descartes invented the system that has become the foundation of algebra while sick in bed. According to the story, Descartes was staring at a fly crawling on the ceiling when he realized that he could describe the fly's location in relation to the perpendicular lines formed by the adjacent walls of his room. He viewed the perpendicular lines as horizontal and vertical axes. Further, by dividing each axis into equal unit lengths, Descartes saw that it was possible to locate any object in a two-dimensional plane using just two numbers—the displacement from the horizontal axis and the displacement from the vertical axis.

While there is evidence that ideas similar to Descartes' grid system existed centuries earlier, it was Descartes who introduced the components that comprise the **Cartesian coordinate system**, a grid system having perpendicular axes. Descartes named the horizontal axis the ***x*-axis** and the vertical axis the ***y*-axis**.

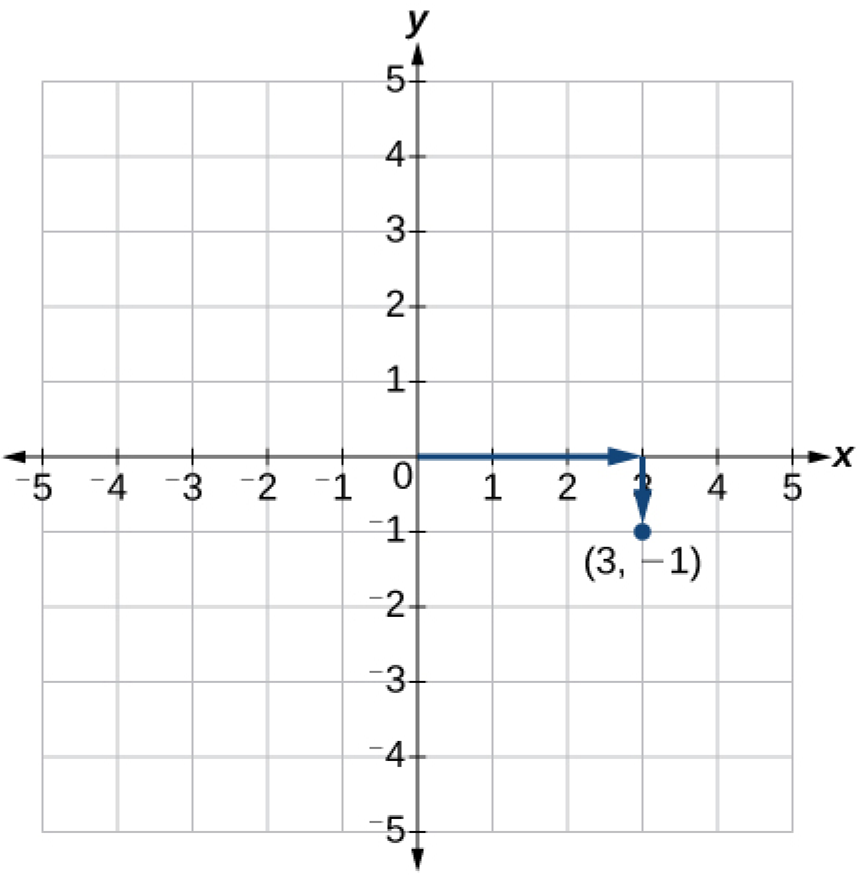
The Cartesian coordinate system, also called the rectangular coordinate system, is based on a two-dimensional plane consisting of the *x*-axis and the *y*-axis. Perpendicular to each other, the axes divide the plane into four sections. Each section is called a **quadrant**; the quadrants are numbered counterclockwise as shown in [link]


y-axis
Quadrant II
Quadrant I
x-axis
Quadrant III
Quadrant IV

The center of the plane is the point at which the two axes cross. It is known as the **origin**, or point$(0,0)$.From the origin, each axis is further divided into equal units: increasing, positive numbers to the right on the *x*-axis and up the *y*-axis; decreasing, negative numbers to the left on the *x*-axis and down the *y*-axis. The axes extend to positive and negative infinity as shown by the arrowheads in [link].



Each point in the plane is identified by its ***x*-coordinate**, or horizontal displacement from the origin, and its ***y*-coordinate**, or vertical displacement from the origin. Together, we write them as an **ordered pair** indicating the combined distance from the origin in the form (x, y). An ordered pair is also known as a coordinate pair because it consists of *x*- and *y*-coordinates. For example, we can represent the point $(3, -1)$ in the plane by moving three units to the right of the origin in the horizontal direction, and one unit down in the vertical direction. See [link].



When dividing the axes into equally spaced increments, note that the *x*-axis may be considered separately from the *y*-axis. In other words, while the *x*-axis may be divided and labeled according to consecutive integers, the *y*-axis may be divided and labeled by increments of 2, or 10, or 100. In fact, the axes may represent other units, such as years against the balance in a savings account, or quantity against cost, and so on. Consider the rectangular coordinate system primarily as a method for showing the relationship between two quantities.

Note:
Cartesian Coordinate System
A two-dimensional plane where the

- *x*-axis is the horizontal axis
- *y*-axis is the vertical axis

A point in the plane is defined as an ordered pair, (x, y),such that *x* is determined by its horizontal distance from the origin and *y* is determined by its vertical distance from the origin.

Example:
Exercise:

Problem:
Plotting Points in a Rectangular Coordinate System

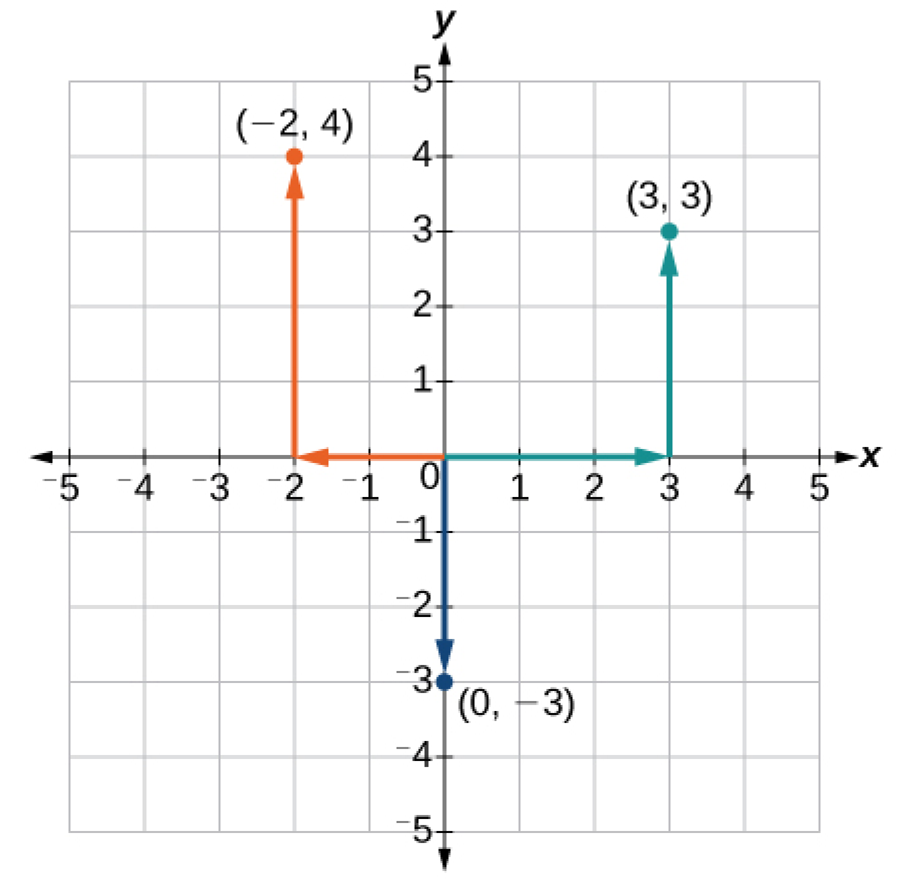
Plot the points $(-2, 4)$,$(3, 3)$,and $(0, -3)$ in the plane.

Solution:

To plot the point $(-2, 4)$,begin at the origin. The *x*-coordinate is –2, so move two units to the left. The *y*-coordinate is 4, so then move four units up in the positive *y* direction.

To plot the point $(3, 3)$,begin again at the origin. The *x*-coordinate is 3, so move three units to the right. The *y*-coordinate is also 3, so move three units up in the positive *y* direction.

To plot the point $(0, -3)$,begin again at the origin. The *x*-coordinate is 0. This tells us not to move in either direction along the *x*-axis. The *y*-coordinate is –3, so move three units down in the negative *y* direction. See the graph in [link].



Analysis

Note that when either coordinate is zero, the point must be on an axis. If the *x*-coordinate is zero, the point is on the *y*-axis. If the *y*-coordinate is zero, the point is on the *x*-axis.

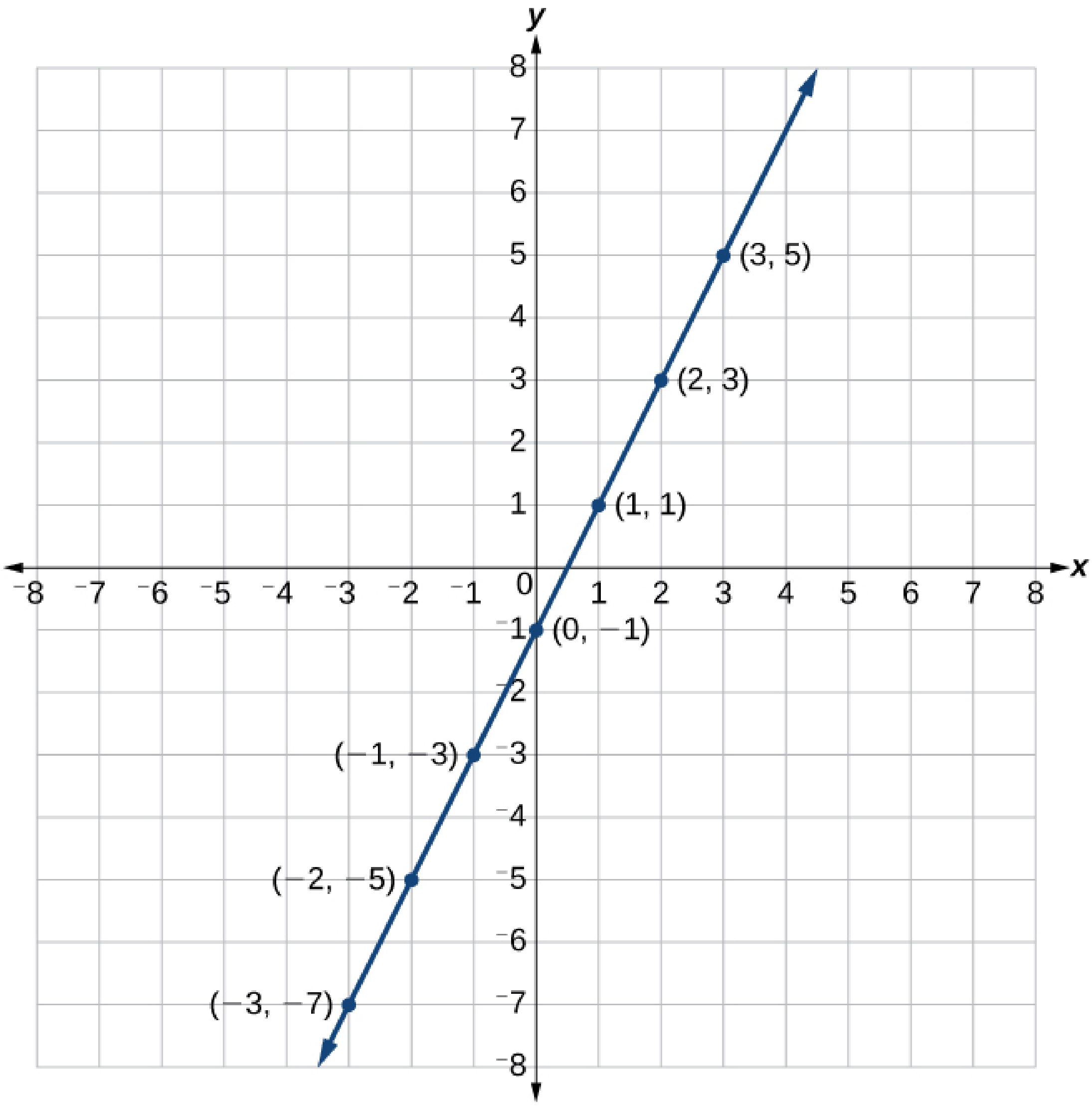
Graphing Equations by Plotting Points

We can plot a set of points to represent an equation. When such an equation contains both an *x* variable and a *y* variable, it is called an **equation in two variables**. Its graph is called a **graph in two variables**. Any graph on a two-dimensional plane is a graph in two variables.

Suppose we want to graph the equation $y = 2x - 1$. We can begin by substituting a value for *x* into the equation and determining the resulting value of *y*. Each pair of *x*- and *y*-values is an ordered pair that can be plotted. [link] lists values of *x* from –3 to 3 and the resulting values for *y*.

x	$y = 2x - 1$	(x, y)
-3	$y = 2(-3) - 1 = -7$	$(-3, -7)$
-2	$y = 2(-2) - 1 = -5$	$(-2, -5)$
-1	$y = 2(-1) - 1 = -3$	$(-1, -3)$
0	$y = 2(0) - 1 = -1$	$(0, -1)$
1	$y = 2(1) - 1 = 1$	$(1, 1)$
2	$y = 2(2) - 1 = 3$	$(2, 3)$
3	$y = 2(3) - 1 = 5$	$(3, 5)$

We can plot the points in the table. The points for this particular equation form a line, so we can connect them. See [link]**.** This is not true for all equations.



Note that the *x*-values chosen are arbitrary, regardless of the type of equation we are graphing. Of course, some situations may require particular values of *x* to be plotted in order to see a particular result. Otherwise, it is logical to choose values that can be calculated easily, and it is always a good idea to choose values that are both negative and positive. There is no rule dictating how many points to plot, although we need at least two to graph a line. Keep in mind, however, that the more points we plot, the more accurately we can sketch the graph.

Note:

Given an equation, graph by plotting points.

1. Make a table with one column labeled *x*, a second column labeled with the equation, and a third column listing the resulting ordered pairs.
2. Enter *x*-values down the first column using positive and negative values. Selecting the *x*-values in numerical order will make the graphing simpler.
3. Select *x*-values that will yield *y*-values with little effort, preferably ones that can be calculated mentally.
4. Plot the ordered pairs.
5. Connect the points if they form a line.

Example:
Exercise:

Problem:
Graphing an Equation in Two Variables by Plotting Points

Graph the equation $y=-x+2$ by plotting points.

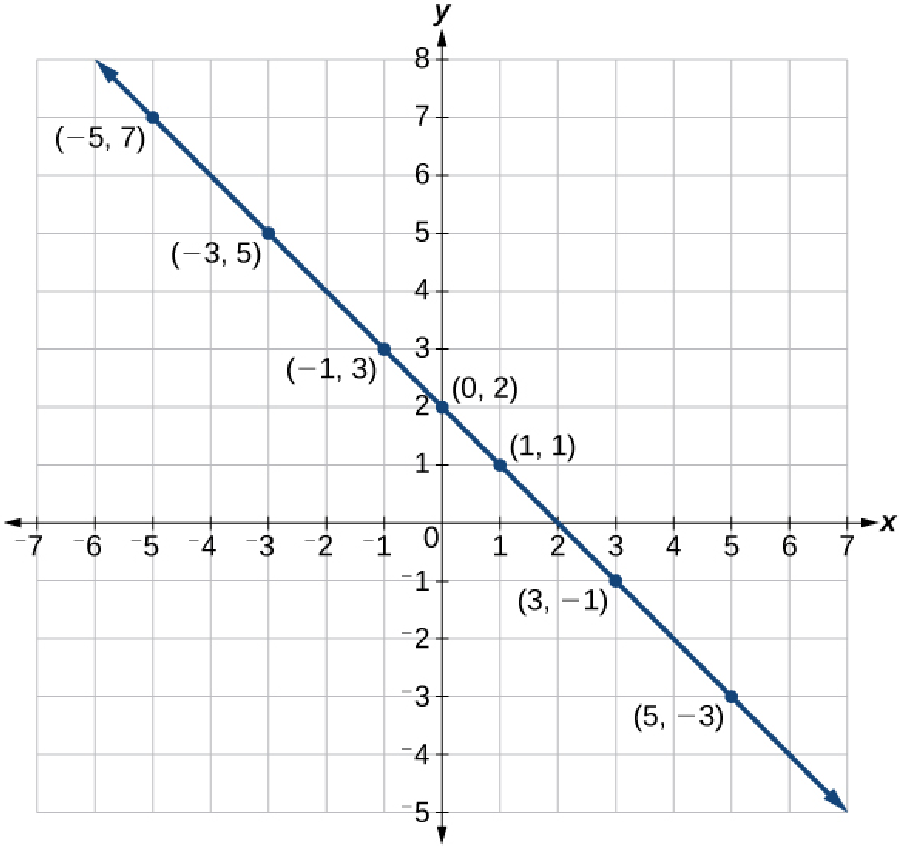
Solution:

First, we construct a table similar to [link]. Choose *x* values and calculate *y*.

x	$y=-x+2$	(x,y)
-5	$y=-(-5)+2=7$	$(-5,7)$

-3	$y = -(-3) + 2 = 5$	$(-3, 5)$
-1	$y = -(-1) + 2 = 3$	$(-1, 3)$
0	$y = -(0) + 2 = 2$	$(0, 2)$
1	$y = -(1) + 2 = 1$	$(1, 1)$
3	$y = -(3) + 2 = -1$	$(3, -1)$
5	$y = -(5) + 2 = -3$	$(5, -3)$

Now, plot the points. Connect them if they form a line. See [link]



Note:

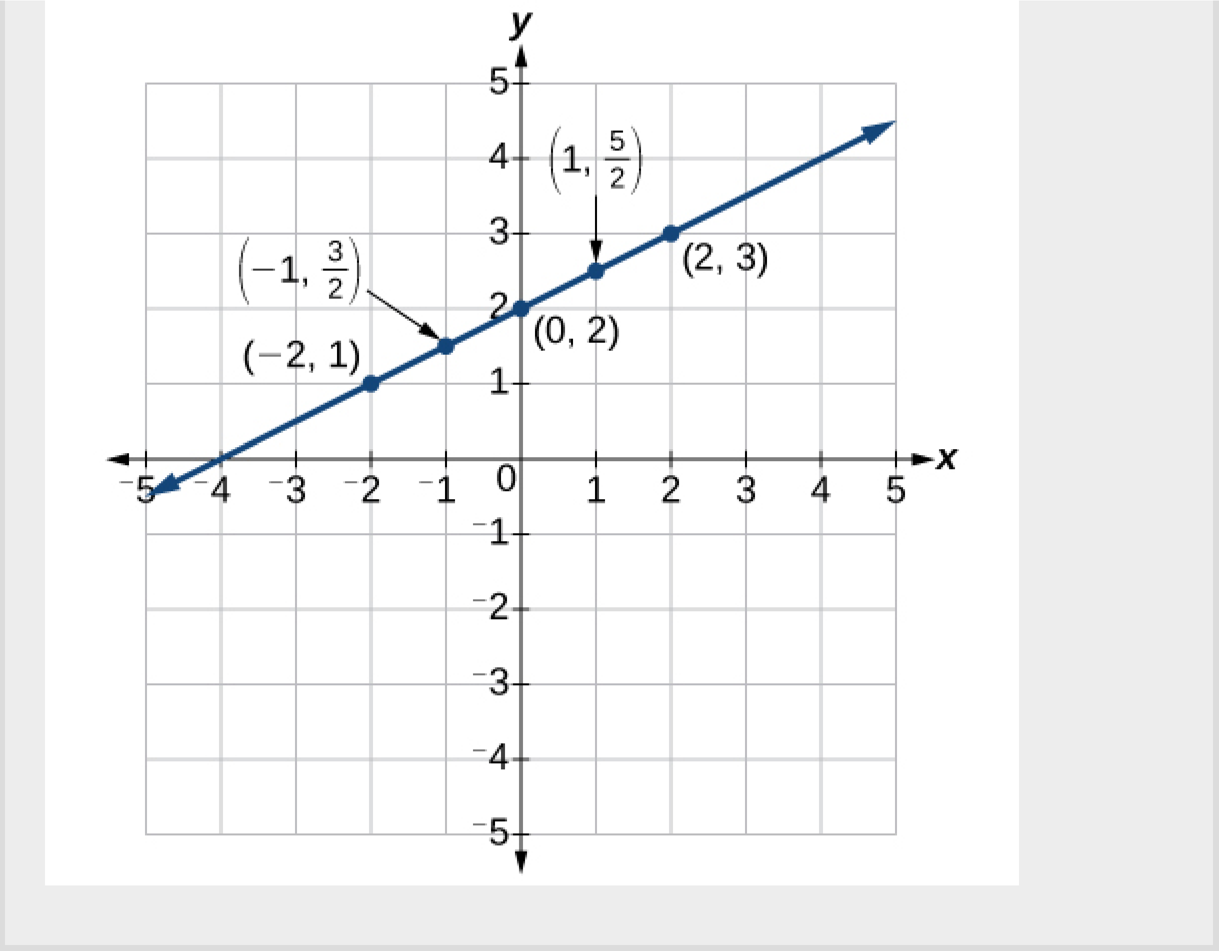
Exercise:

Problem:

Construct a table and graph the equation by plotting points: $y = \frac{1}{2}x + 2$.

Solution:

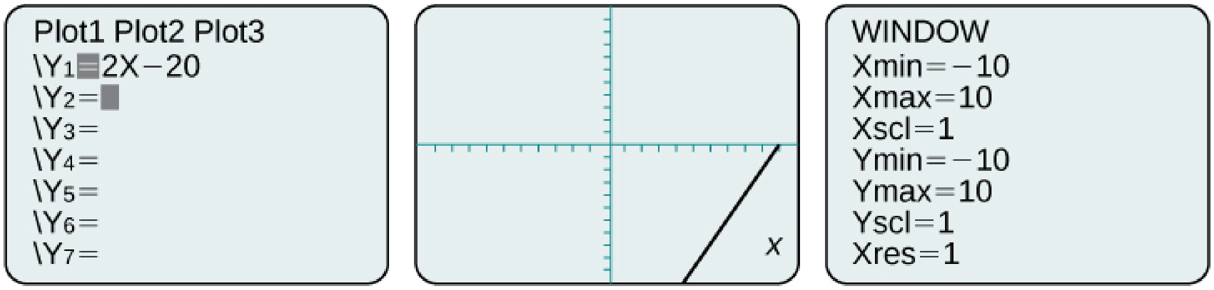
x	$y = \frac{1}{2}x + 2$	(x, y)
-2	$y = \frac{1}{2}(-2) + 2 = 1$	$(-2, 1)$
-1	$y = \frac{1}{2}(-1) + 2 = \frac{3}{2}$	$\left(-1, \frac{3}{2}\right)$
0	$y = \frac{1}{2}(0) + 2 = 2$	$(0, 2)$
1	$y = \frac{1}{2}(1) + 2 = \frac{5}{2}$	$\left(1, \frac{5}{2}\right)$
2	$y = \frac{1}{2}(2) + 2 = 3$	$(2, 3)$



Graphing Equations with a Graphing Utility

Most graphing calculators require similar techniques to graph an equation. The equations sometimes have to be manipulated so they are written in the style $y =$______. The TI-84 Plus, and many other calculator makes and models, have a mode function, which allows the window (the screen for viewing the graph) to be altered so the pertinent parts of a graph can be seen.

For example, the equation $y = 2x - 20$ has been entered in the TI-84 Plus shown in [link]**a.** In [link]**b,** the resulting graph is shown. Notice that we cannot see on the screen where the graph crosses the axes. The standard window screen on the TI-84 Plus shows $-10 \leq x \leq 10$,and $-10 \leq y \leq 10$. See [link]**c**.



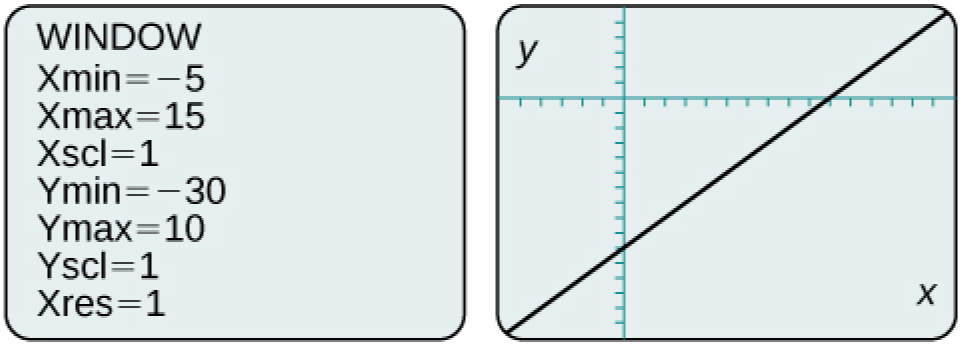
Plot1 Plot2 Plot3
\Y1=2X−20
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=

WINDOW
Xmin=−10
Xmax=10
Xscl=1
Ymin=−10
Ymax=10
Yscl=1
Xres=1

(a) (b) (c)

a. Enter the equation. b. This is the graph in the original window. c. These are the original settings.

By changing the window to show more of the positive *x*-axis and more of the negative *y*-axis, we have a much better view of the graph and the *x*- and *y*-intercepts. See [link]**a** and [link]**b.**



WINDOW
Xmin=−5
Xmax=15
Xscl=1
Ymin=−30
Ymax=10
Yscl=1
Xres=1

(a) (b)

a. This screen shows the new window settings. b. We can clearly view the intercepts in the new window.

Example:
Exercise:

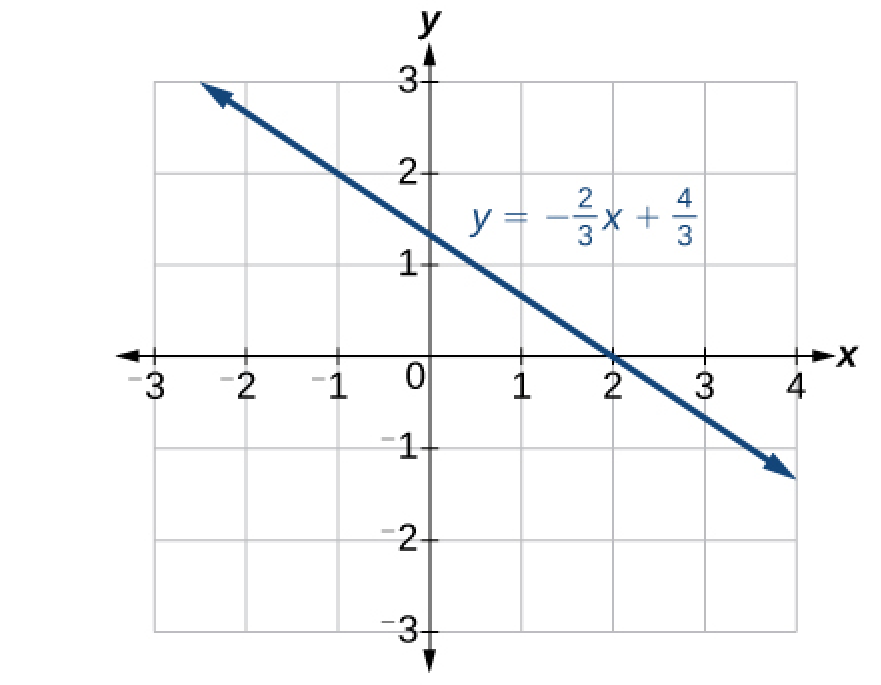
Problem:

Using a Graphing Utility to Graph an Equation

Use a graphing utility to graph the equation: $y = -\frac{2}{3}x - \frac{4}{3}$.

Solution:

Enter the equation in the *y*= function of the calculator. Set the window settings so that both the *x*- and *y*- intercepts are showing in the window. See [link].



Finding *x*-intercepts and *y*-intercepts

The **intercepts** of a graph are points at which the graph crosses the axes. The ***x*-intercept** is the point at which the graph crosses the *x*-axis. At this point, the *y*-coordinate is zero. The ***y*-intercept** is the point at which the graph crosses the *y*-axis. At this point, the *x*-coordinate is zero.

To determine the *x*-intercept, we set *y* equal to zero and solve for *x*. Similarly, to determine the *y*-intercept, we set *x* equal to zero and solve for *y*. For example, lets find the intercepts of the equation $y = 3x - 1$.

To find the *x*-intercept, set $y = 0$.

Equation:

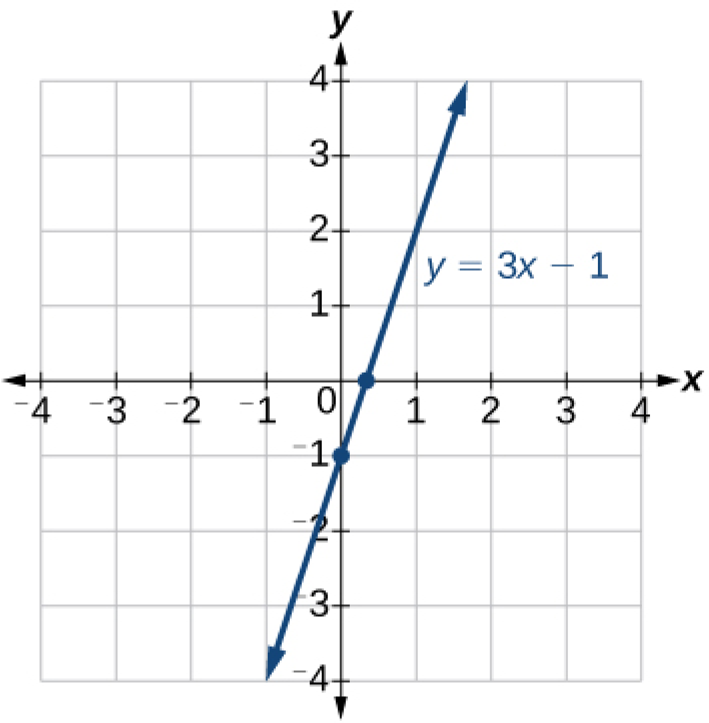
$$\begin{array}{ll} y = 3x - 1 & \\ 0 = 3x - 1 & \\ 1 = 3x & \\ \frac{1}{3} = x & \\ \left(\frac{1}{3}, 0\right) & \qquad x\text{–intercept} \end{array}$$

To find the *y*-intercept, set $x = 0$.

Equation:

$$\begin{array}{ll} y = 3x - 1 & \\ y = 3(0) - 1 & \\ y = -1 & \\ (0, -1) & \qquad y\text{–intercept} \end{array}$$

We can confirm that our results make sense by observing a graph of the equation as in [link]. Notice that the graph crosses the axes where we predicted it would.



Note:
Given an equation, find the intercepts.

- Find the *x*-intercept by setting $y = 0$ and solving for x.
- Find the *y*-intercept by setting $x = 0$ and solving for y.

Example:
Exercise:

Problem:
Finding the Intercepts of the Given Equation

Find the intercepts of the equation $y = -3x - 4$. Then sketch the graph using only the intercepts.

Solution:

Set $y = 0$ to find the *x*-intercept.

Equation:

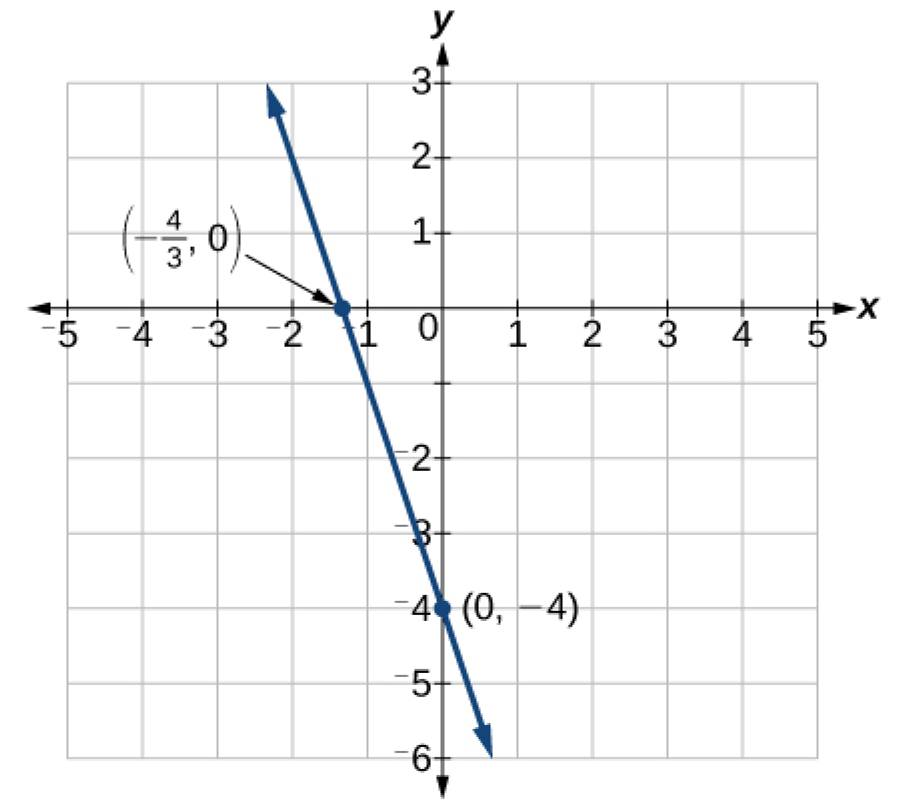
$$\begin{aligned} y &= -3x - 4 \\ 0 &= -3x - 4 \\ 4 &= -3x \\ -\tfrac{4}{3} &= x \\ \left(-\tfrac{4}{3}, 0\right) & \qquad x\text{−intercept} \end{aligned}$$

Set $x = 0$ to find the *y*-intercept.

Equation:

$$\begin{aligned} & y = -3x - 4 \\ & y = -3(0) - 4 \\ & y = -4 \\ & (0, -4) \qquad y\text{−intercept} \end{aligned}$$

Plot both points, and draw a line passing through them as in [link].



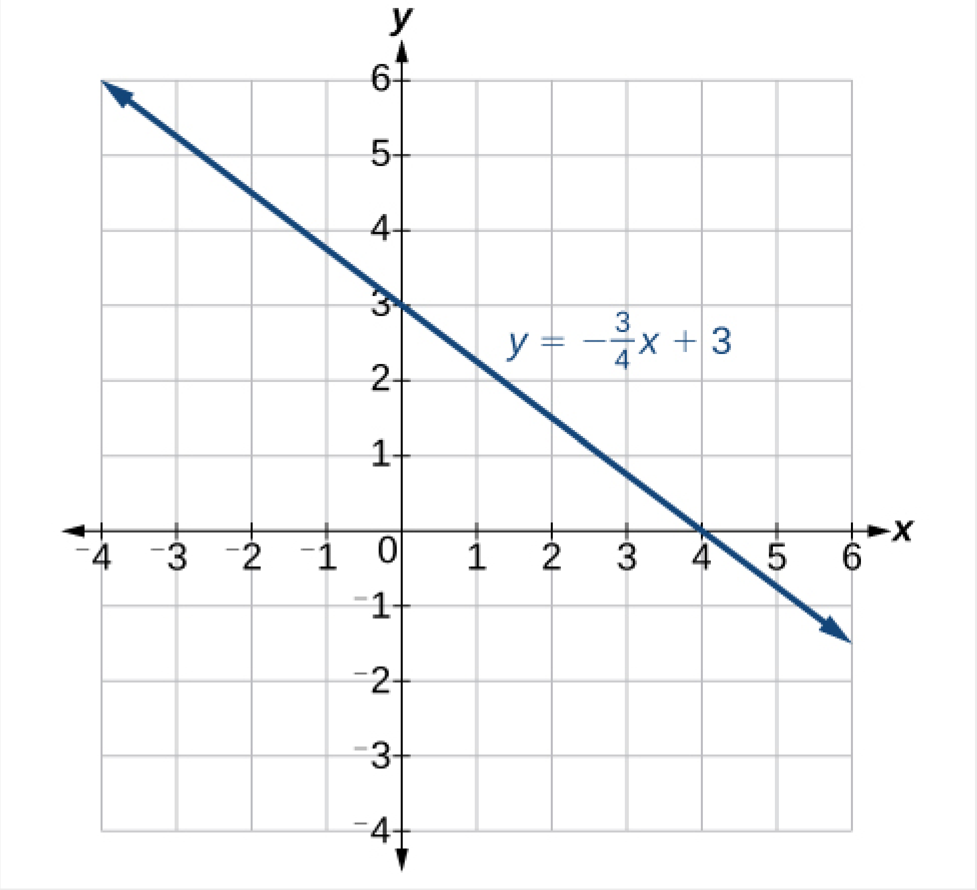
Note:
Exercise:

Problem:

Find the intercepts of the equation and sketch the graph: $y = -\frac{3}{4}x + 3.$

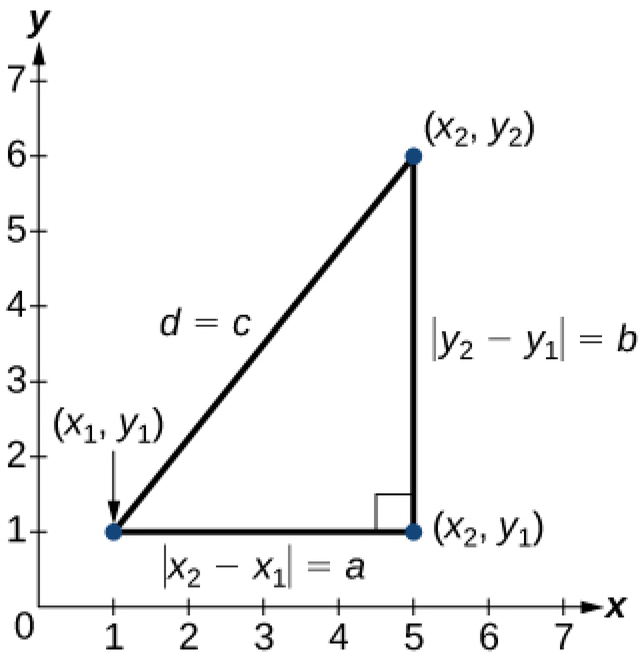
Solution:

x-intercept is $(4, 0)$;*y*-intercept is $(0, 3)$.



Using the Distance Formula

Derived from the Pythagorean Theorem, the **distance formula** is used to find the distance between two points in the plane. The Pythagorean Theorem, $a^2 + b^2 = c^2$,is based on a right triangle where a and b are the lengths of the legs adjacent to the right angle, and c is the length of the hypotenuse. See [link].



The relationship of sides $|x_2 - x_1|$ and $|y_2 - y_1|$ to side *d* is the same as that of sides *a* and *b* to side *c*. We use the absolute value symbol to indicate that the length is a positive number because the absolute value of any number is positive. (For example, $|-3| = 3.$) The symbols $|x_2 - x_1|$ and $|y_2 - y_1|$ indicate that the lengths of the sides of the triangle are positive. To find the length *c*, take the square root of both sides of the Pythagorean Theorem.

Equation:

$$c^2 = a^2 + b^2 \rightarrow c = \sqrt{a^2 + b^2}$$

It follows that the distance formula is given as

Equation:

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \rightarrow d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We do not have to use the absolute value symbols in this definition because any number squared is positive.

Note:
The Distance Formula
Given endpoints (x_1, y_1) and (x_2, y_2),the distance between two points is given by
Equation:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

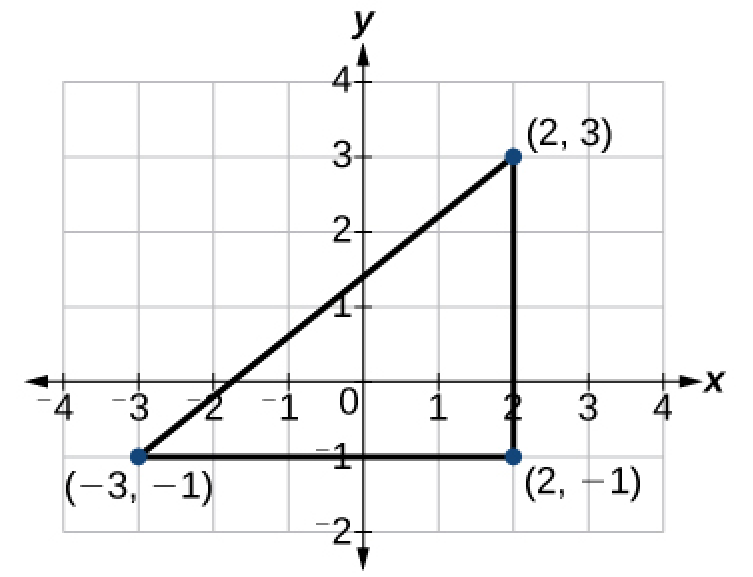
Example:
Exercise:

Problem:
Finding the Distance between Two Points

Find the distance between the points $(-3, -1)$ and $(2, 3)$.

Solution:

Let us first look at the graph of the two points. Connect the points to form a right triangle as in [link].



Then, calculate the length of *d* using the distance formula.

Equation:

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ d &= \sqrt{(2 - (-3))^2 + (3 - (-1))^2} \\ &= \sqrt{(5)^2 + (4)^2} \\ &= \sqrt{25 + 16} \\ &= \sqrt{41} \end{aligned}$$

Note:
Exercise:

Problem: Find the distance between two points: $(1, 4)$ and $(11, 9)$.

Solution:

$\sqrt{125} = 5\sqrt{5}$

Example:
Exercise:

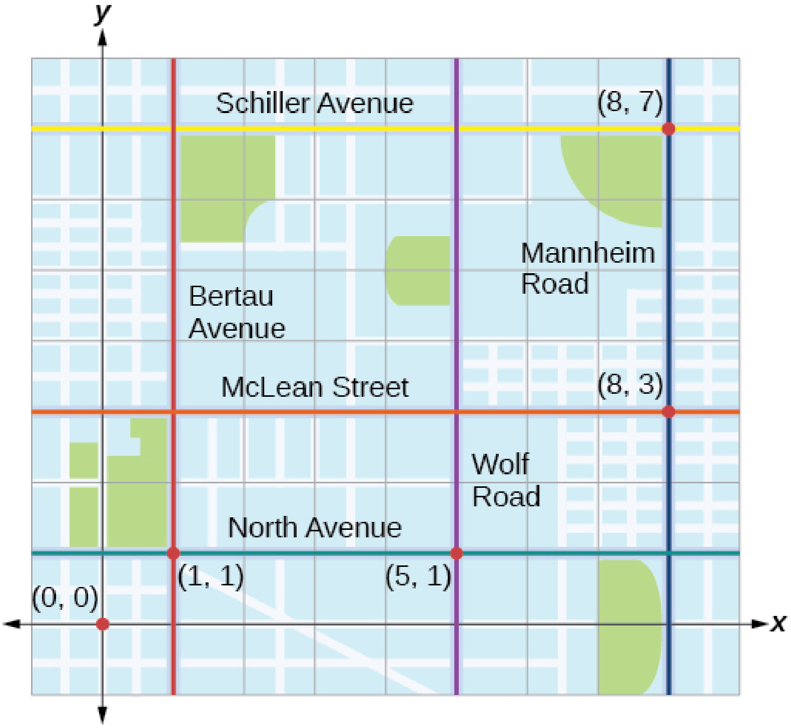
Problem:
Finding the Distance between Two Locations

Let’s return to the situation introduced at the beginning of this section.

Tracie set out from Elmhurst, IL, to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot in [link]. Find the total distance that Tracie traveled. Compare this with the distance between her starting and final positions.

Solution:

The first thing we should do is identify ordered pairs to describe each position. If we set the starting position at the origin, we can identify each of the other points by counting units east (right) and north (up) on the grid. For example, the first stop is 1 block east and 1 block north, so it is at $(1,1)$. The next stop is 5 blocks to the east, so it is at $(5,1)$. After that, she traveled 3 blocks east and 2 blocks north to $(8,3)$. Lastly, she traveled 4 blocks north to $(8,7)$. We can label these points on the grid as in [link].


y
Schiller Avenue
(8, 7)
Mannheim Road
Bertau Avenue
McLean Street
(8, 3)
Wolf Road
North Avenue
(1, 1)
(5, 1)
(0, 0)
x

Next, we can calculate the distance. Note that each grid unit represents 1,000 feet.

- From her starting location to her first stop at $(1,1)$,Tracie might have driven north 1,000 feet and then east 1,000 feet, or vice versa. Either way, she drove 2,000 feet to her first stop.

- Her second stop is at $(5, 1)$. So from $(1, 1)$ to $(5, 1)$,Tracie drove east 4,000 feet.
- Her third stop is at $(8, 3)$. There are a number of routes from $(5, 1)$ to $(8, 3)$. Whatever route Tracie decided to use, the distance is the same, as there are no angular streets between the two points. Let's say she drove east 3,000 feet and then north 2,000 feet for a total of 5,000 feet.
- Tracie's final stop is at $(8, 7)$. This is a straight drive north from $(8, 3)$ for a total of 4,000 feet.

Next, we will add the distances listed in [link].

From/To	**Number of Feet Driven**
$(0, 0)$ to $(1, 1)$	2,000
$(1, 1)$ to$(5, 1)$	4,000
$(5, 1)$ to $(8, 3)$	5,000
$(8, 3)$ to $(8, 7)$	4,000
Total	15,000

The total distance Tracie drove is 15,000 feet, or 2.84 miles. This is not, however, the actual distance between her starting and ending positions. To find this distance, we can use the distance formula between the points $(0, 0)$ and $(8, 7)$.

Equation:

$$\begin{aligned} d &= \sqrt{(8-0)^2 + (7-0)^2} \\ &= \sqrt{64+49} \\ &= \sqrt{113} \\ &\approx 10.63 \text{ units} \end{aligned}$$

At 1,000 feet per grid unit, the distance between Elmhurst, IL, to Franklin Park is 10,630.14 feet, or 2.01 miles. The distance formula results in a shorter calculation because it is based on the hypotenuse of a right triangle, a straight diagonal from the origin to the point $(8, 7)$. Perhaps you have heard the saying "as the crow flies," which means the shortest distance between two points because a crow can fly in a straight line even though a person on the ground has to travel a longer distance on existing roadways.

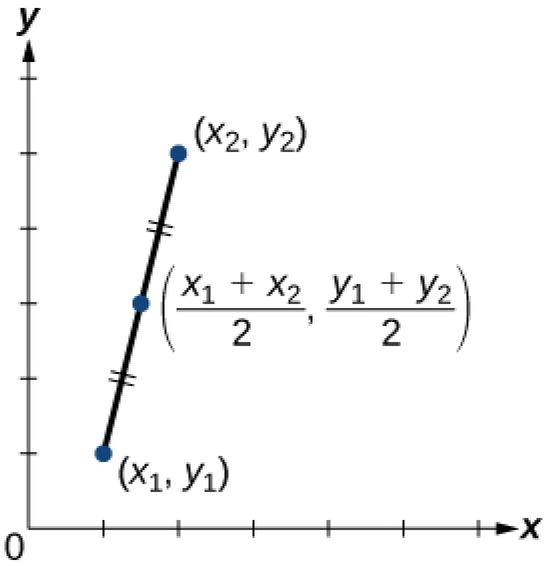
Using the Midpoint Formula

When the endpoints of a line segment are known, we can find the point midway between them. This point is known as the midpoint and the formula is known as the **midpoint formula**. Given the endpoints of a line segment, (x_1, y_1) and (x_2, y_2),the midpoint formula states how to find the coordinates of the midpoint M.

Equation:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

A graphical view of a midpoint is shown in [link]. Notice that the line segments on either side of the midpoint are congruent.


y
(x_2, y_2)
$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
(x_1, y_1)
0
x

Example:
Exercise:

Problem:
Finding the Midpoint of the Line Segment

Find the midpoint of the line segment with the endpoints $(7, -2)$ and $(9, 5)$.

Solution:

Use the formula to find the midpoint of the line segment.
Equation:

$$\begin{aligned}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) &= \left(\frac{7+9}{2}, \frac{-2+5}{2}\right) \\ &= \left(8, \frac{3}{2}\right)\end{aligned}$$

Note:
Exercise:

Problem:

Find the midpoint of the line segment with endpoints $(-2,-1)$ and $(-8,6)$.

Solution:

$\left(-5,\frac{5}{2}\right)$

Example:
Exercise:

Problem:
Finding the Center of a Circle

The diameter of a circle has endpoints $(-1,-4)$ and $(5,-4)$. Find the center of the circle.

Solution:

The center of a circle is the center, or midpoint, of its diameter. Thus, the midpoint formula will yield the center point.
Equation:

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$
$$\left(\frac{-1+5}{2},\frac{-4-4}{2}\right)=\left(\frac{4}{2},-\frac{8}{2}\right)=(2,-4)$$

Note:
Access these online resources for additional instruction and practice with the Cartesian coordinate system.

- Plotting points on the coordinate plane
- Find x and y intercepts based on the graph of a line

Key Concepts

- We can locate, or plot, points in the Cartesian coordinate system using ordered pairs, which are defined as displacement from the *x*-axis and displacement from the *y*-axis. See [link].
- An equation can be graphed in the plane by creating a table of values and plotting points. See [link].
- Using a graphing calculator or a computer program makes graphing equations faster and more accurate. Equations usually have to be entered in the form $y=$_____. See [link].
- Finding the *x*- and *y*-intercepts can define the graph of a line. These are the points where the graph crosses the axes. See [link].
- The distance formula is derived from the Pythagorean Theorem and is used to find the length of a line segment. See [link] and [link].
- The midpoint formula provides a method of finding the coordinates of the midpoint dividing the sum of the *x*-coordinates and the sum of the *y*-coordinates of the endpoints by 2. See [link] and [link].

Section Exercises

Verbal

Exercise:

Problem:

Is it possible for a point plotted in the Cartesian coordinate system to not lie in one of the four quadrants? Explain.

Solution:

Answers may vary. Yes. It is possible for a point to be on the *x*-axis or on the *y*-axis and therefore is considered to NOT be in one of the quadrants.

Exercise:

Problem:

Describe the process for finding the *x*-intercept and the *y*-intercept of a graph algebraically.

Exercise:

Problem:

Describe in your own words what the *y*-intercept of a graph is.

Solution:

The *y*-intercept is the point where the graph crosses the *y*-axis.

Exercise:

Problem:

When using the distance formula $d=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$, explain the correct order of operations that are to be performed to obtain the correct answer.

Algebraic

For each of the following exercises, find the *x*-intercept and the *y*-intercept without graphing. Write the coordinates of each intercept.

Exercise:

Problem:$y=-3x+6$

Solution:

The *x*-intercept is $(2,0)$ and the *y*-intercept is $(0,6)$.

Exercise:

Problem:$4y=2x-1$

Exercise:

Problem:$3x-2y=6$

Solution:

The *x*-intercept is $(2,0)$ and the *y*-intercept is $(0,-3)$.

Exercise:

Problem:$4x-3=2y$

Exercise:

Problem:$3x+8y=9$

Solution:

The *x*-intercept is $(3,0)$ and the *y*-intercept is $\left(0,\frac{9}{8}\right)$.

Exercise:

Problem:$2x-\frac{2}{3}=\frac{3}{4}y+3$

For each of the following exercises, solve the equation for *y* in terms of *x*.

Exercise:

Problem:$4x+2y=8$

Solution:

$y=4-2x$

Exercise:

Problem:$3x - 2y = 6$

Exercise:

Problem:$2x = 5 - 3y$

Solution:

$y = \frac{5-2x}{3}$

Exercise:

Problem:$x - 2y = 7$

Exercise:

Problem:$5y + 4 = 10x$

Solution:

$y = 2x - \frac{4}{5}$

Exercise:

Problem:$5x + 2y = 0$

For each of the following exercises, find the distance between the two points. Simplify your answers, and write the exact answer in simplest radical form for irrational answers.

Exercise:

Problem:$(-4, 1)$ and $(3, -4)$

Solution:

$d=\sqrt{74}$

Exercise:

Problem: $(2,-5)$ and $(7,4)$

Exercise:

Problem: $(5,0)$ and $(5,6)$

Solution:

$d=\sqrt{36}=6$

Exercise:

Problem: $(-4,3)$ and $(10,3)$

Exercise:

Problem:

Find the distance between the two points given using your calculator, and round your answer to the nearest hundredth.

$(19,12)$ and $(41,71)$

Solution:

$d\approx 62.97$

For each of the following exercises, find the coordinates of the midpoint of the line segment that joins the two given points.

Exercise:

Problem: $(-5,-6)$ and $(4,2)$

Exercise:

Problem:$(-1, 1)$ and $(7, -4)$

Solution:

$(3, \frac{-3}{2})$

Exercise:

Problem:$(-5, -3)$ and $(-2, -8)$

Exercise:

Problem:$(0, 7)$ and $(4, -9)$

Solution:

$(2, -1)$

Exercise:

Problem:$(-43, 17)$ and $(23, -34)$

Graphical

For each of the following exercises, identify the information requested.

Exercise:

Problem: What are the coordinates of the origin?

Solution:

$(0, 0)$

Exercise:

Problem: If a point is located on the *y*-axis, what is the *x*-coordinate?

Exercise:

Problem: If a point is located on the *x*-axis, what is the *y*-coordinate?

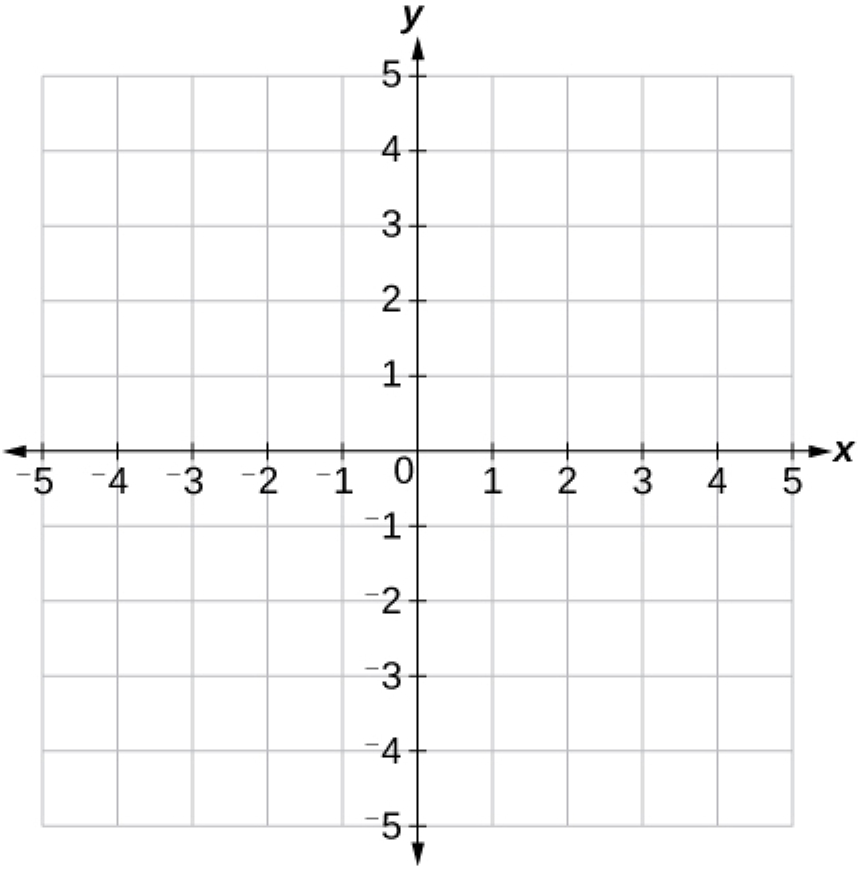
Solution:

$y = 0$

For each of the following exercises, plot the three points on the given coordinate plane. State whether the three points you plotted appear to be collinear (on the same line).

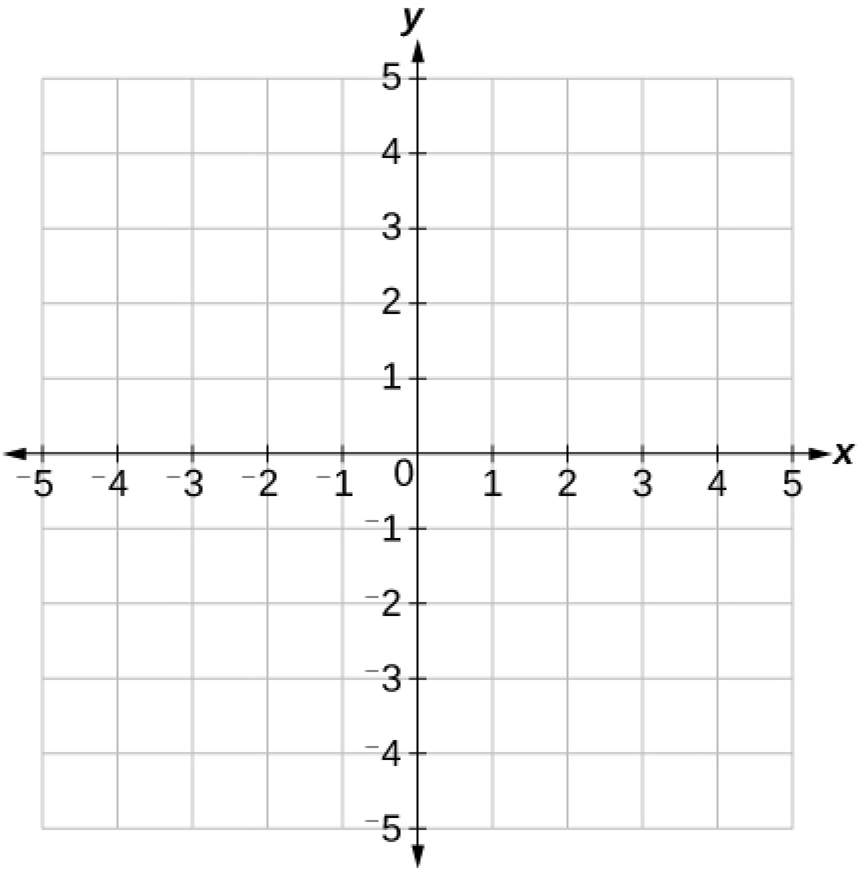
Exercise:

Problem:$(4, 1)$ $(-2, -3)$ $(5, 0)$

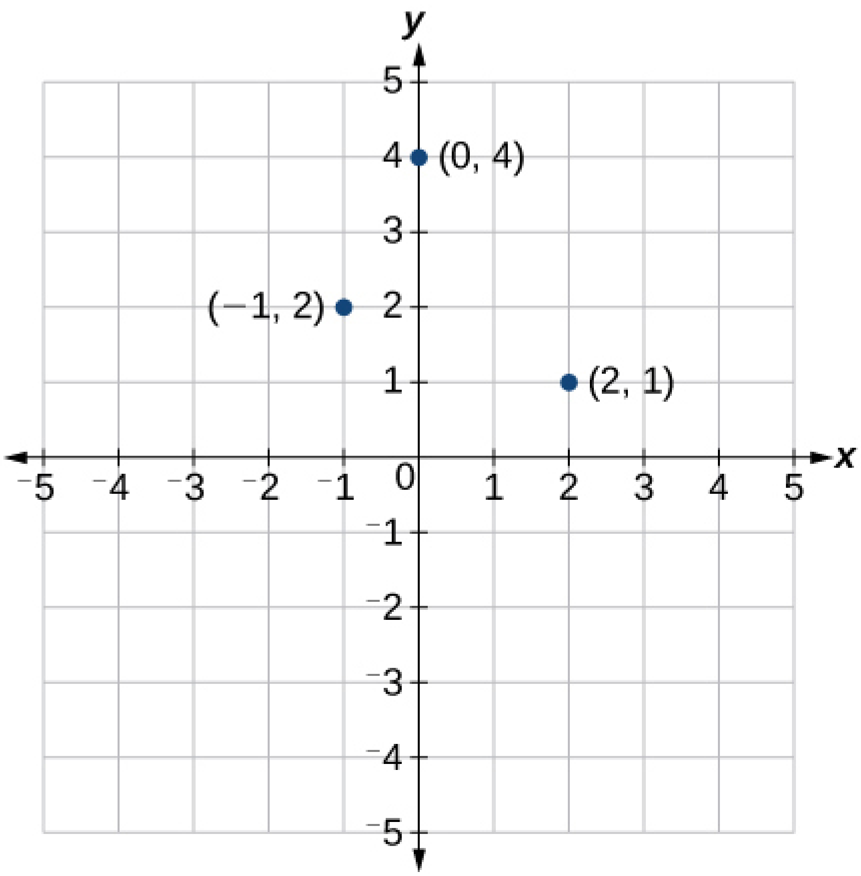


Exercise:

Problem: $(-1, 2)(0, 4)(2, 1)$



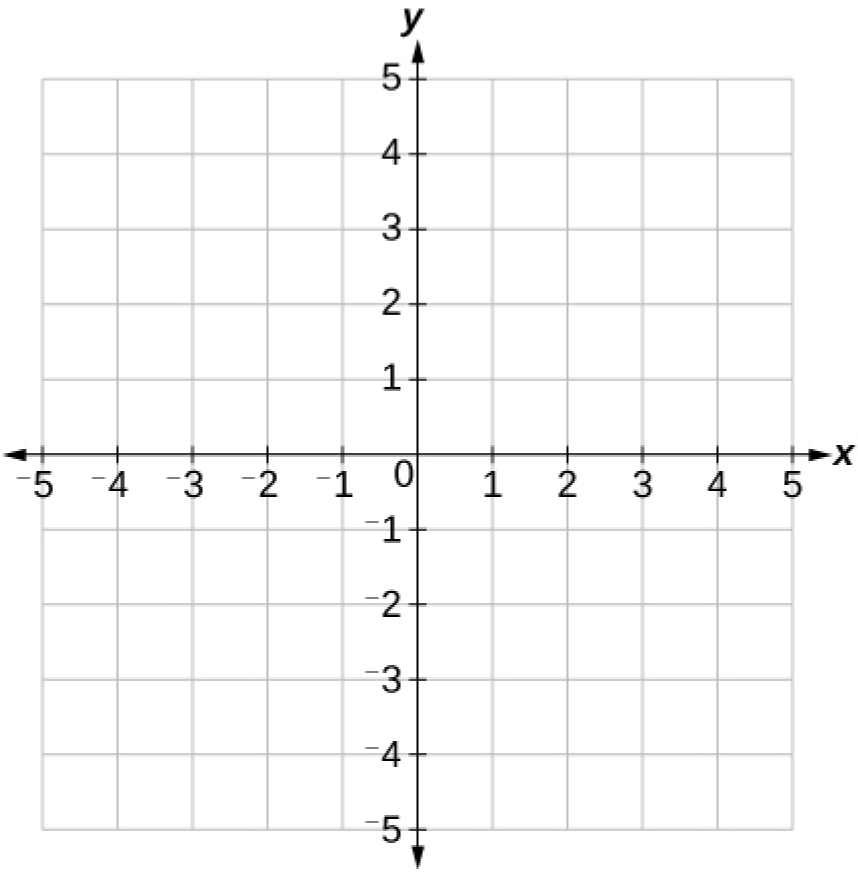
Solution:



not collinear

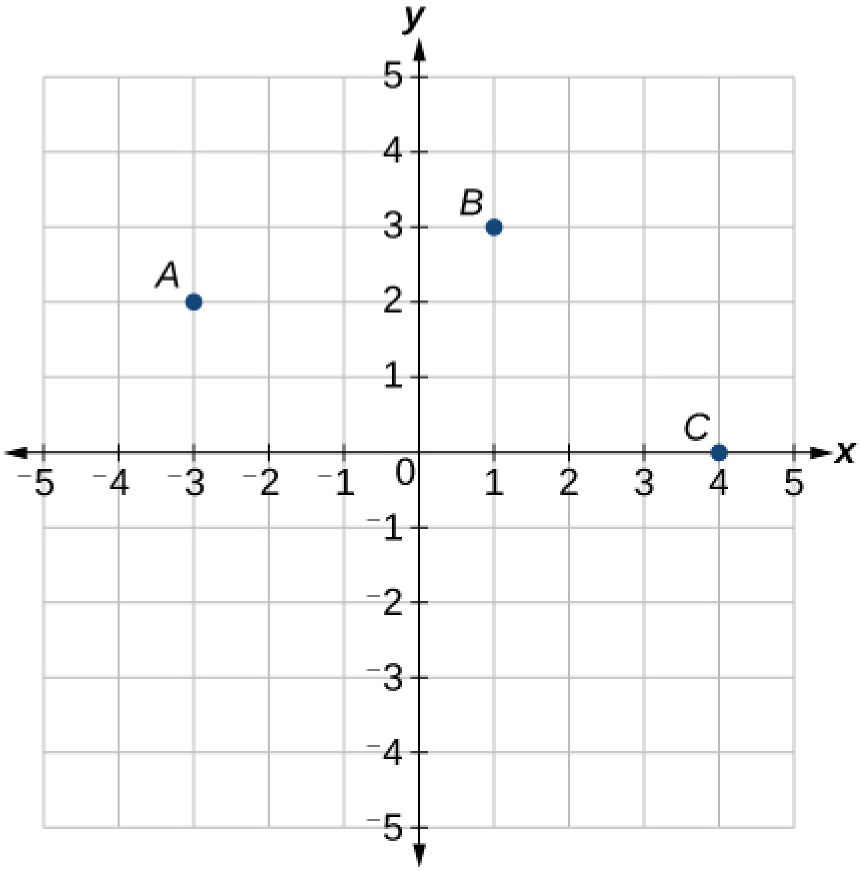
Exercise:

Problem: $(-3, 0)$ $(-3, 4)$ $(-3, -3)$


y
x
5
4
3
2
1
0
−1
−2
−3
−4
−5
1
2
3
4
5
−1
−2
−3
−4
−5

Exercise:

Problem: Name the coordinates of the points graphed.



Solution:

$(-3, 2), (1, 3), (4, 0)$

Exercise:

Problem:

Name the quadrant in which the following points would be located. If the point is on an axis, name the axis.

$a. (-3, -4)$
$b. (-5, 0)$
$c. (1, -4)$
$d. (-2, 7)$
$e. (0, -3)$

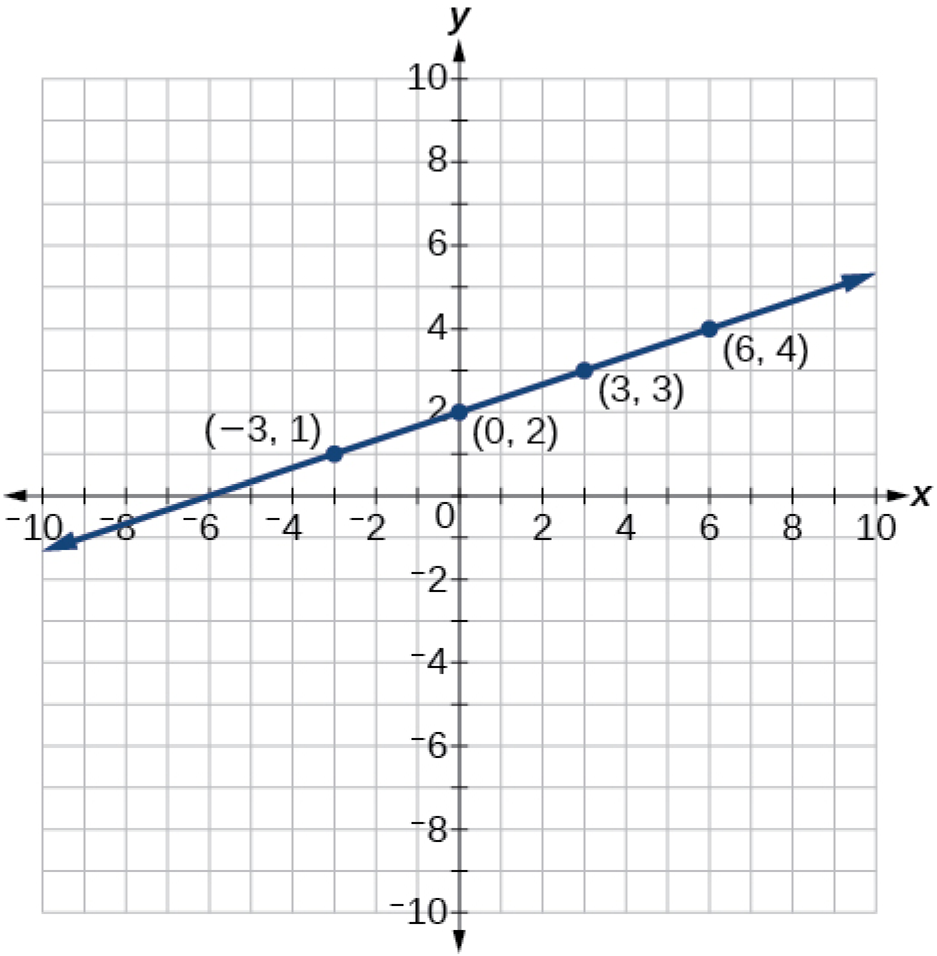
For each of the following exercises, construct a table and graph the equation by plotting at least three points.

Exercise:

Problem: $y = \frac{1}{3}x + 2$

Solution:

x	y
-3	1
0	2
3	3
6	4



Exercise:

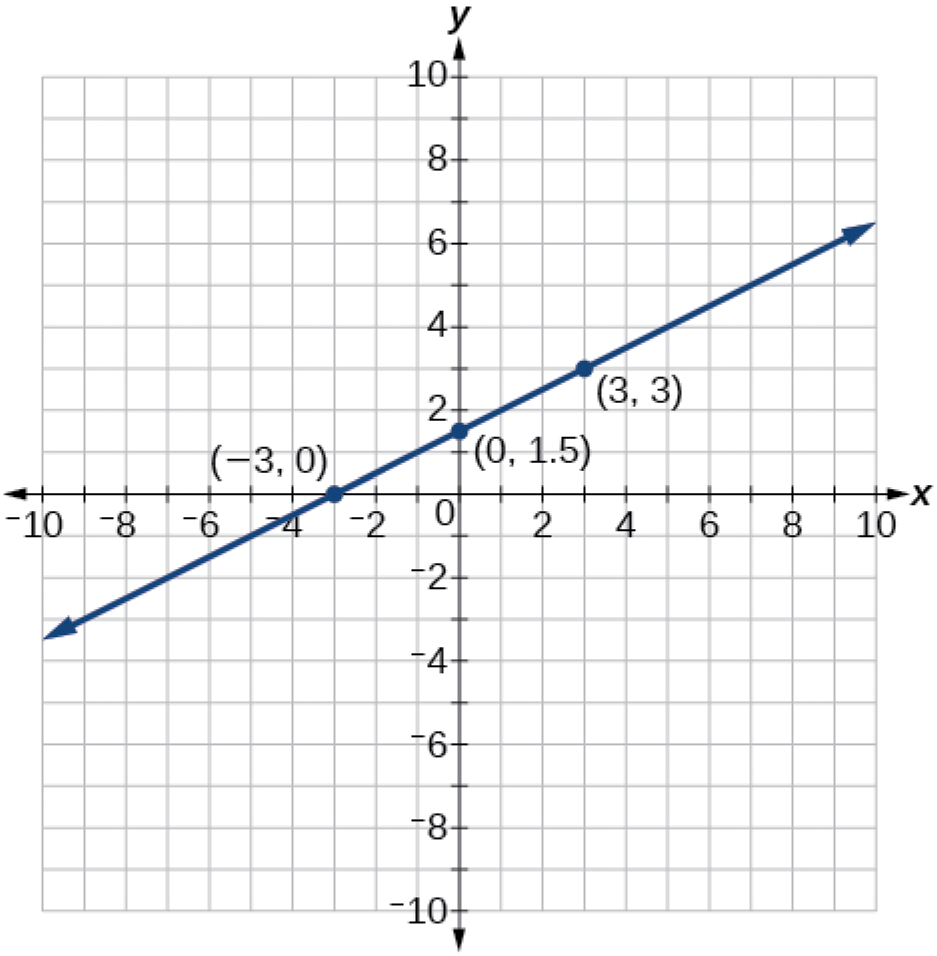
Problem:$y = -3x + 1$

Exercise:

Problem:$2y = x + 3$

Solution:

x	*y*
–3	0
0	1.5
3	3



Numeric

For each of the following exercises, find and plot the *x*- and *y*-intercepts, and graph the straight line based on those two points.

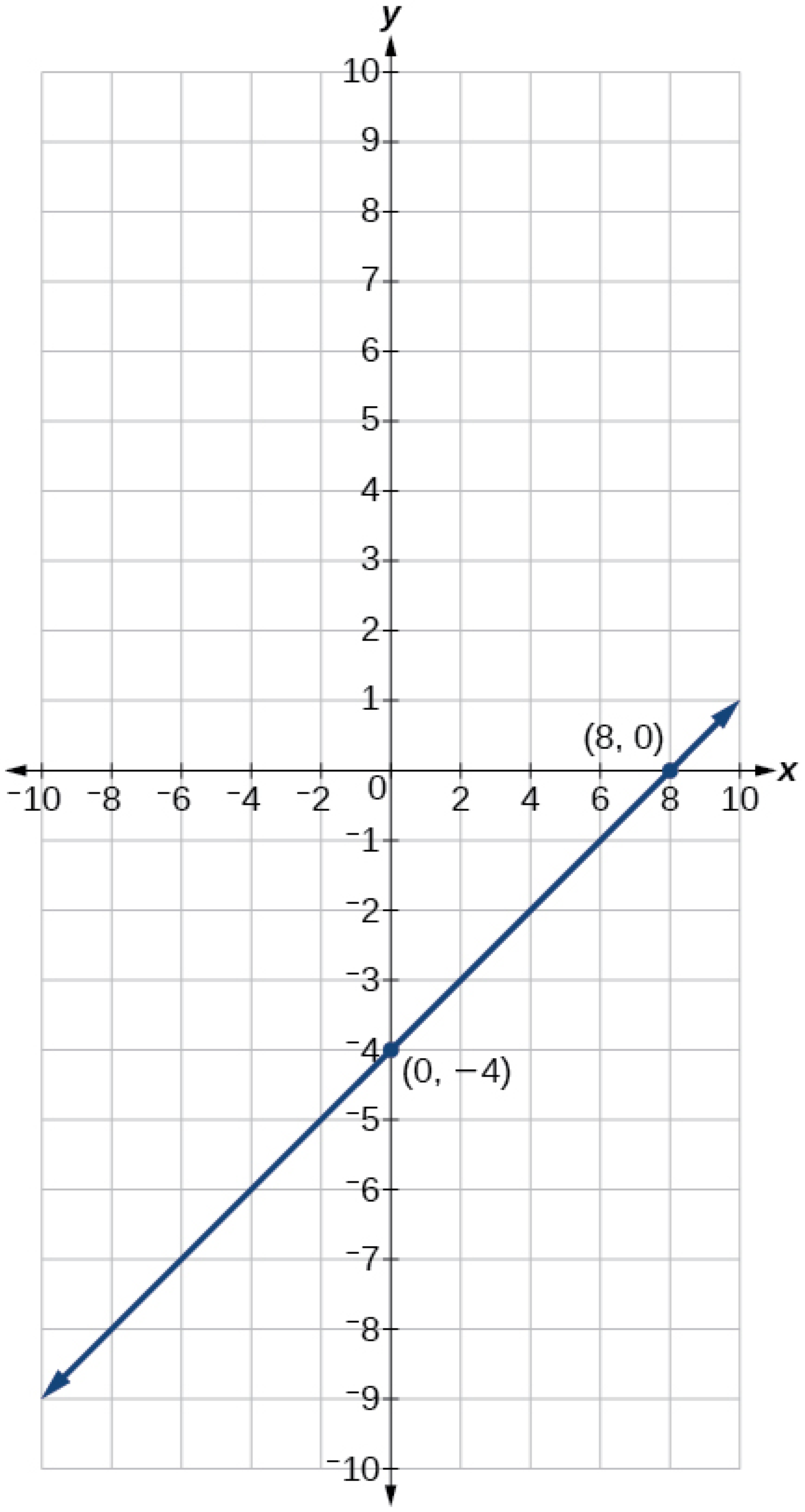
Exercise:

Problem: $4x - 3y = 12$

Exercise:

Problem: $x - 2y = 8$

Solution:


y
x
(8, 0)
(0, −4)
10
9
8
7
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−7
−8
−9
−10
−10
−8
−6
−4
−2
2
4
6
8
10

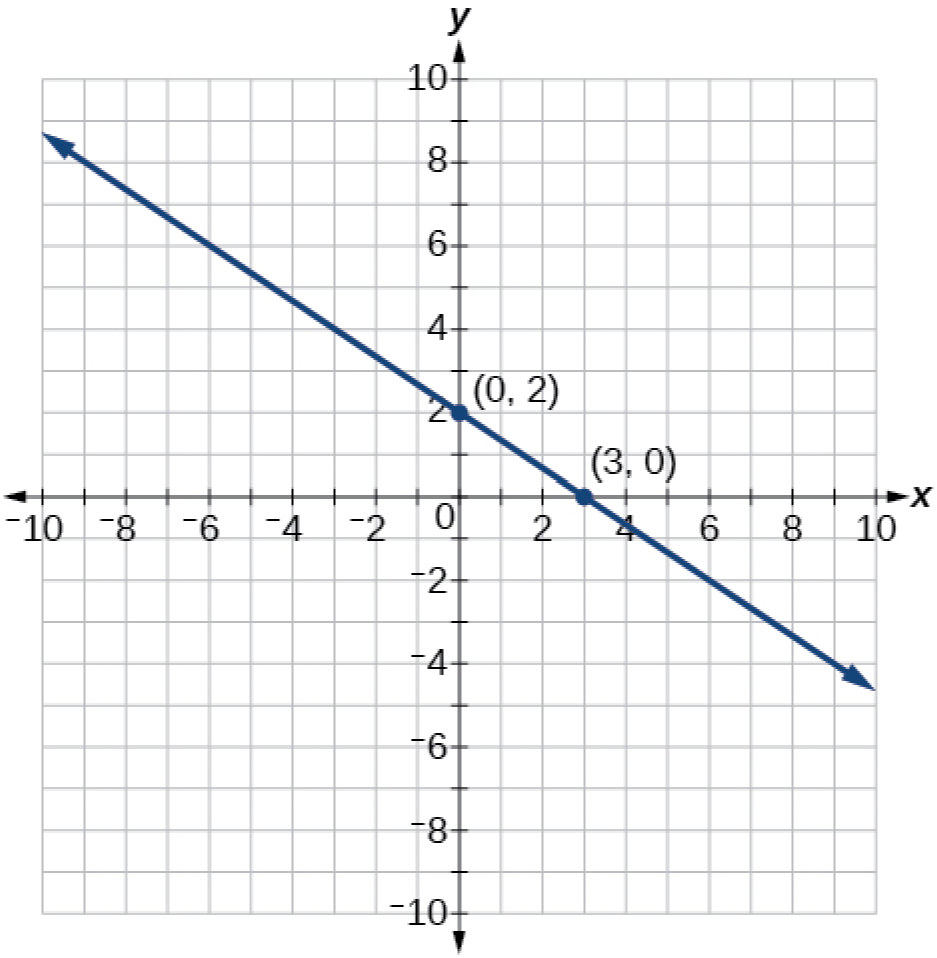
Exercise:

Problem: $y - 5 = 5x$

Exercise:

Problem: $3y = -2x + 6$

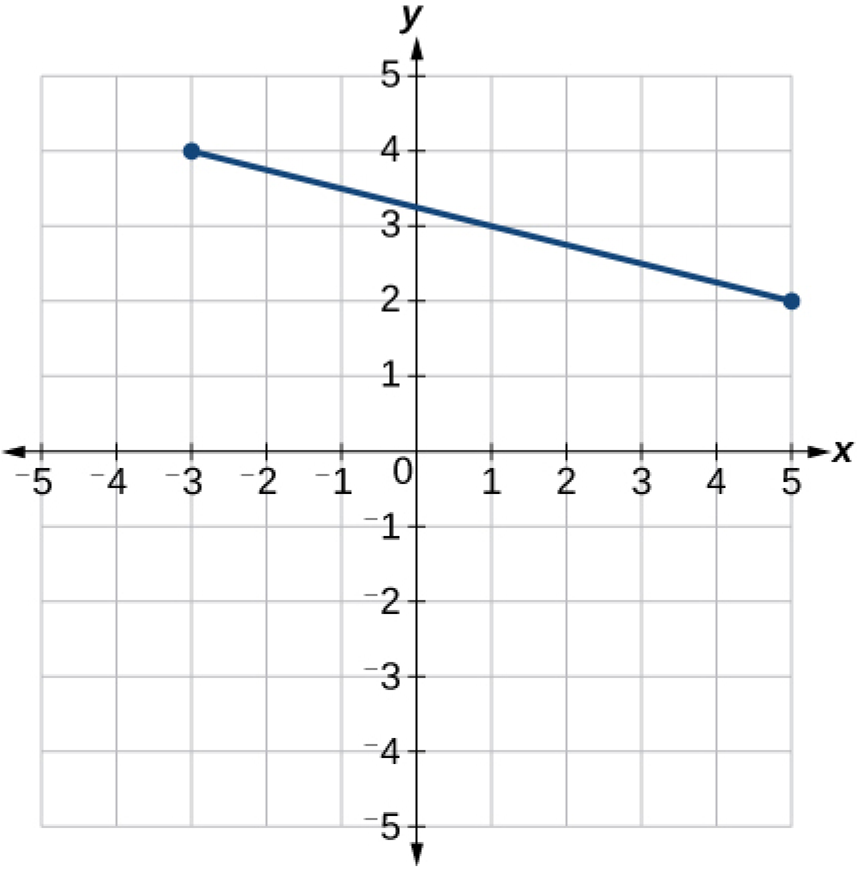
Solution:



Exercise:

Problem: $y = \frac{x-3}{2}$

For each of the following exercises, use the graph in the figure below.



Exercise:

Problem:

Find the distance between the two endpoints using the distance formula. Round to three decimal places.

Solution:

$d = 8.246$

Exercise:

Problem:

Find the coordinates of the midpoint of the line segment connecting the two points.

Exercise:

Problem: Find the distance that $(-3,4)$ is from the origin.

Solution:

$d=5$

Exercise:

Problem:

Find the distance that $(5,2)$ is from the origin. Round to three decimal places.

Exercise:

Problem: Which point is closer to the origin?

Solution:

$(-3,4)$

Technology

For the following exercises, use your graphing calculator to input the linear graphs in the Y= graph menu.

After graphing it, use the 2^{nd} CALC button and 1:value button, hit enter. At the lower part of the screen you will see "x=" and a blinking cursor. You may enter any number for *x* and it will display the *y* value for any *x* value you input. Use this and plug in *x* = 0, thus finding the *y*-intercept, for each of the following graphs.

Exercise:

Problem: $\mathrm{Y}_1=-2x+5$

Exercise:

Problem: $\mathrm{Y}_1 = \frac{3x-8}{4}$

Solution:

$x = 0 \qquad y = -2$

Exercise:

Problem: $\mathrm{Y}_1 = \frac{x+5}{2}$

For the following exercises, use your graphing calculator to input the linear graphs in the Y= graph menu.

After graphing it, use the 2nd CALC button and 2:zero button, hit enter. At the lower part of the screen you will see “left bound?” and a blinking cursor on the graph of the line. Move this cursor to the left of the *x*-intercept, hit ENTER. Now it says “right bound?” Move the cursor to the right of the *x*-intercept, hit enter. Now it says “guess?” Move your cursor to the left somewhere in between the left and right bound near the *x*-intercept. Hit enter. At the bottom of your screen it will display the coordinates of the *x*-intercept or the “zero” to the *y*-value. Use this to find the *x*-intercept.

Note: With linear/straight line functions the zero is not really a “guess,” but it is necessary to enter a “guess” so it will search and find the exact *x*-intercept between your right and left boundaries. With other types of functions (more than one *x*-intercept), they may be irrational numbers so “guess” is more appropriate to give it the correct limits to find a very close approximation between the left and right boundaries.

Exercise:

Problem: $\mathrm{Y}_1 = -8x + 6$

Solution:

$x = 0.75 \quad y = 0$

Exercise:

Problem:$\mathrm{Y}_1 = 4x - 7$

Exercise:

Problem:$\mathrm{Y}_1 = \frac{3x+5}{4}$ Round your answer to the nearest thousandth.

Solution:

$x = -1.667 \quad y = 0$

Extensions

Exercise:

Problem:

A man drove 10 mi directly east from his home, made a left turn at an intersection, and then traveled 5 mi north to his place of work. If a road was made directly from his home to his place of work, what would its distance be to the nearest tenth of a mile?

Exercise:

Problem:

If the road was made in the previous exercise, how much shorter would the man's one-way trip be every day?

Solution:

$15{-}11.2 \;=\; 3.8$ mi shorter

Exercise:

Problem:

Given these four points: $A\left(1,3\right), B\left(-3,5\right), C\left(4,7\right),\ \text{and}\ \ D\left(5,-4\right)$,find the coordinates of the midpoint of line segments AB and CD.

Exercise:

Problem:

After finding the two midpoints in the previous exercise, find the distance between the two midpoints to the nearest thousandth.

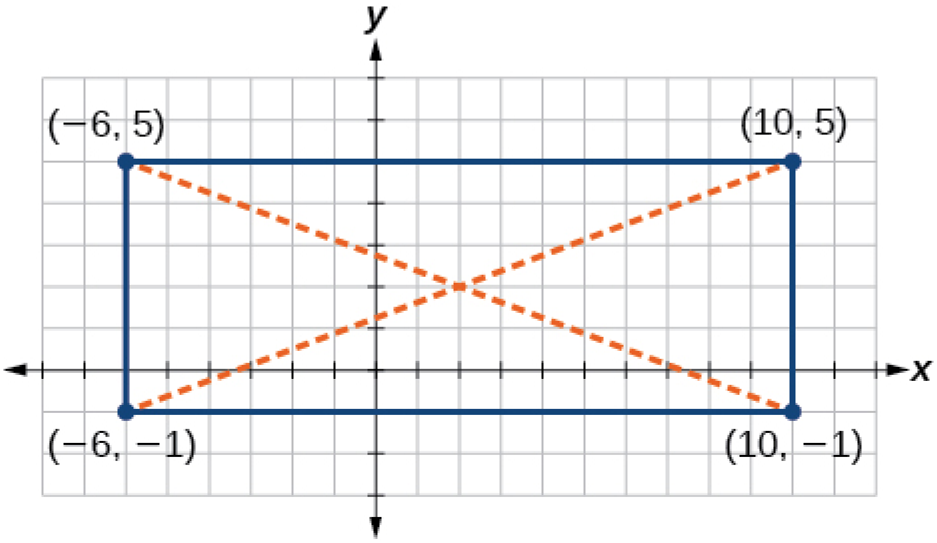
Solution:

6.042

Exercise:

Problem:

Given the graph of the rectangle shown and the coordinates of its vertices, prove that the diagonals of the rectangle are of equal length.



Exercise:

Problem:

In the previous exercise, find the coordinates of the midpoint for each diagonal.

Solution:

Midpoint of each diagonal is the same point $(2,2)$. Note this is a characteristic of rectangles, but not other quadrilaterals.

Real-World Applications

Exercise:

Problem:

The coordinates on a map for San Francisco are $(53,17)$ and those for Sacramento are $(123,78)$. Note that coordinates represent miles. Find the distance between the cities to the nearest mile.

Exercise:

Problem:

If San Jose's coordinates are $(76,-12)$,where the coordinates represent miles, find the distance between San Jose and San Francisco to the nearest mile.

Solution:

37 mi

Exercise:

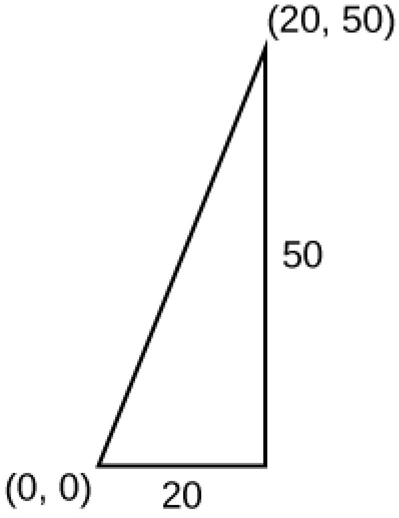
Problem:

A small craft in Lake Ontario sends out a distress signal. The coordinates of the boat in trouble were $(49, 64)$. One rescue boat is at the coordinates $(60, 82)$ and a second Coast Guard craft is at coordinates $(58, 47)$. Assuming both rescue craft travel at the same rate, which one would get to the distressed boat the fastest?

Exercise:

Problem:

A man on the top of a building wants to have a guy wire extend to a point on the ground 20 ft from the building. To the nearest foot, how long will the wire have to be if the building is 50 ft tall?



Solution:

54 ft

Exercise:

Problem:

If we rent a truck and pay a \$75/day fee plus \$.20 for every mile we travel, write a linear equation that would express the total cost y,using x to represent the number of miles we travel. Graph this function on your graphing calculator and find the total cost for one day if we travel 70 mi.

Glossary

Cartesian coordinate system
: a grid system designed with perpendicular axes invented by René Descartes

distance formula
: a formula that can be used to find the length of a line segment if the endpoints are known

equation in two variables
: a mathematical statement, typically written in *x* and *y*, in which two expressions are equal

graph in two variables
: the graph of an equation in two variables, which is always shown in two variables in the two-dimensional plane

intercepts
: the points at which the graph of an equation crosses the *x*-axis and the *y*-axis

midpoint formula
: a formula to find the point that divides a line segment into two parts of equal length

ordered pair
: a pair of numbers indicating horizontal displacement and vertical displacement from the origin; also known as a coordinate pair, (x, y)

origin

the point where the two axes cross in the center of the plane, described by the ordered pair $(0, 0)$

quadrant

one quarter of the coordinate plane, created when the axes divide the plane into four sections

x-axis

the common name of the horizontal axis on a coordinate plane; a number line increasing from left to right

x-coordinate

the first coordinate of an ordered pair, representing the horizontal displacement and direction from the origin

x-intercept

the point where a graph intersects the x-axis; an ordered pair with a y-coordinate of zero

y-axis

the common name of the vertical axis on a coordinate plane; a number line increasing from bottom to top

y-coordinate

the second coordinate of an ordered pair, representing the vertical displacement and direction from the origin

y-intercept

a point where a graph intercepts the y-axis; an ordered pair with an x-coordinate of zero

Use the Rectangular Coordinate System
By the end of this section, you will be able to:

- Plot points on a rectangular coordinate system
- Identify points on a graph
- Verify solutions to an equation in two variables
- Complete a table of solutions to a linear equation
- Find solutions to linear equations in two variables

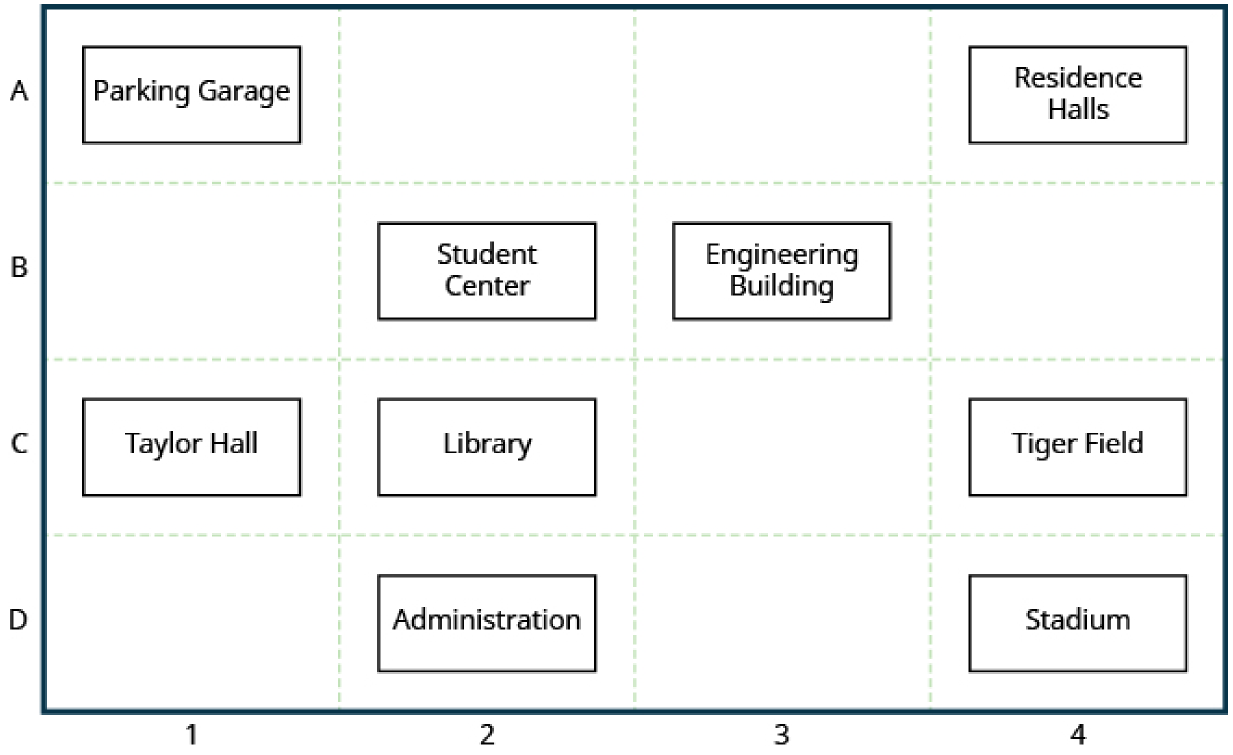
Note:
Before you get started, take this readiness quiz.

1. Evaluate: $x+3$ when $x=-1$.
   If you missed this problem, review [link].
2. Evaluate: $2x-5y$ when $x=3,y=-2$.
   If you missed this problem, review [link].
3. Solve for y: $40-4y=20$.
   If you missed this problem, review [link].

Plot Points on a Rectangular Coordinate System

Many maps, such as the Campus Map shown in [link], use a grid system to identify locations. Do you see the numbers $1,2,3$, and 4 across the top and bottom of the map and the letters A, B, C, and D along the sides? Every location on the map can be identified by a number and a letter.

For example, the Student Center is in section 2B. It is located in the grid section above the number 2 and next to the letter B. In which grid section is the Stadium? The Stadium is in section 4D.


A
Parking Garage
Residence Halls
B
Student Center
Engineering Building
C
Taylor Hall
Library
Tiger Field
D
Administration
Stadium
1
2
3
4

Example:
Exercise:

Problem: Use the map in [link].

ⓐ Find the grid section of the Residence Halls.
ⓑ What is located in grid section 4C?

Solution:
Solution

ⓐ Read the number below the Residence Halls, 4, and the letter to the side, A. So the Residence Halls are in grid section 4A.
ⓑ Find 4 across the bottom of the map and C along the side. Look below the 4 and next to the C. Tiger Field is in grid section 4C.

Note:
Exercise:

Problem: Use the map in [link].

ⓐ Find the grid section of Taylor Hall.
ⓑ What is located in section 3B?

Solution:

ⓐ 1C
ⓑ Engineering Building

Note:
Exercise:

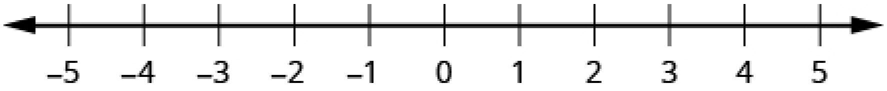
Problem: Use the map in [link].

ⓐ Find the grid section of the Parking Garage.
ⓑ What is located in section 2C?

Solution:

ⓐ 1A
ⓑ Library

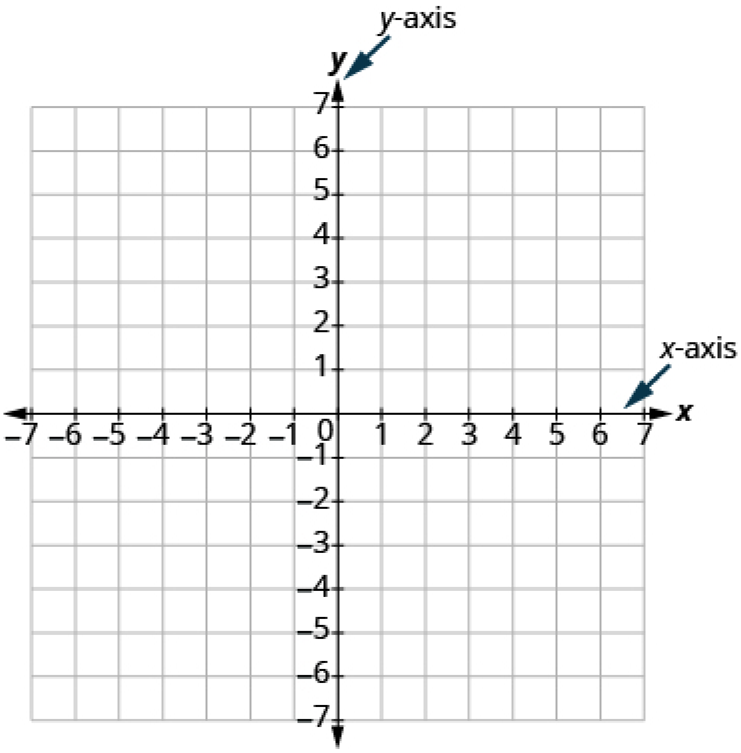
Just as maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a rectangular coordinate system. To create a rectangular coordinate system, start with a horizontal number line. Show both positive and negative numbers as you did before, using a convenient scale unit. This horizontal number line is called the ***x*-axis**.



Now, make a vertical number line passing through the x-axis at 0. Put the positive numbers above 0 and the negative numbers below 0. See [link]. This vertical line is called the ***y*-axis**.

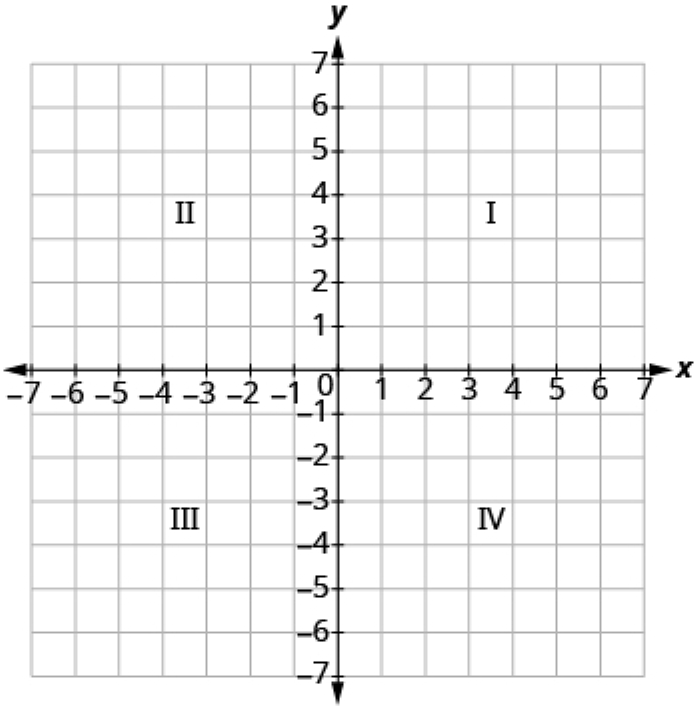
Vertical grid lines pass through the integers marked on the x-axis. Horizontal grid lines pass through the integers marked on the y-axis. The resulting grid is the rectangular coordinate system.

The rectangular coordinate system is also called the x-y plane, the coordinate plane, or the Cartesian coordinate system (since it was developed by a mathematician named René Descartes.)



The rectangular coordinate system.

The x-axis and the y-axis form the rectangular coordinate system. These axes divide a plane into four areas, called **quadrants**. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise. See [link].



The four quadrants of the rectangular coordinate system

In the rectangular coordinate system, every point is represented by an **ordered pair**. The first number in the ordered pair is the *x*-coordinate of the point, and the second number is the *y*-coordinate of the point.

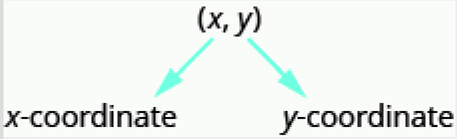
Note:

Ordered Pair

An ordered pair, (x,y) gives the coordinates of a point in a rectangular coordinate system.

Equation:

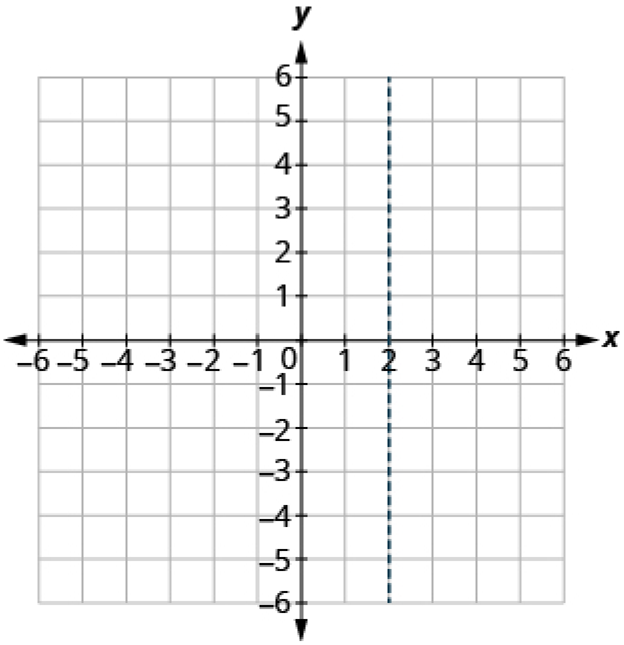
$$\begin{array}{l}\text{The first number is the } x\text{-coordinate.}\\ \text{The second number is the } y\text{-coordinate.}\end{array}$$



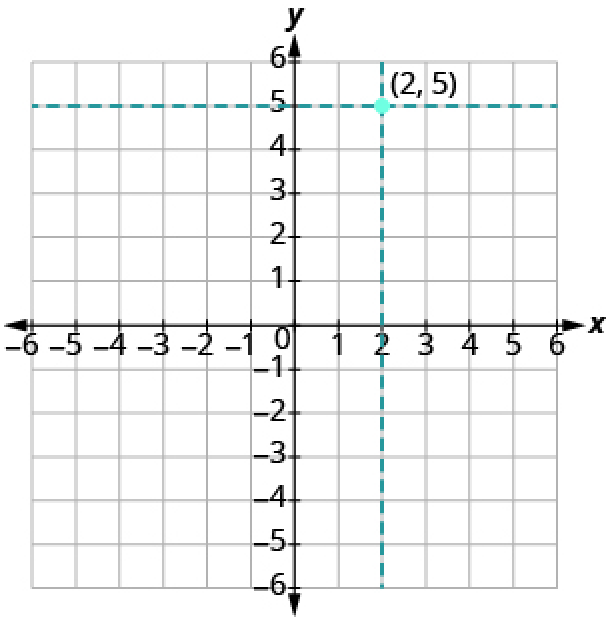
So how do the coordinates of a point help you locate a point on the x-y plane?

Let’s try locating the point $(2,5)$. In this ordered pair, the x-coordinate is 2 and the y-coordinate is 5.

We start by locating the x value, 2, on the x-axis. Then we lightly sketch a vertical line through $x=2$, as shown in [link].



Now we locate the y value, 5, on the y-axis and sketch a horizontal line through $y=5$. The point where these two lines meet is the point with coordinates $(2,5)$. We plot the point there, as shown in [link].



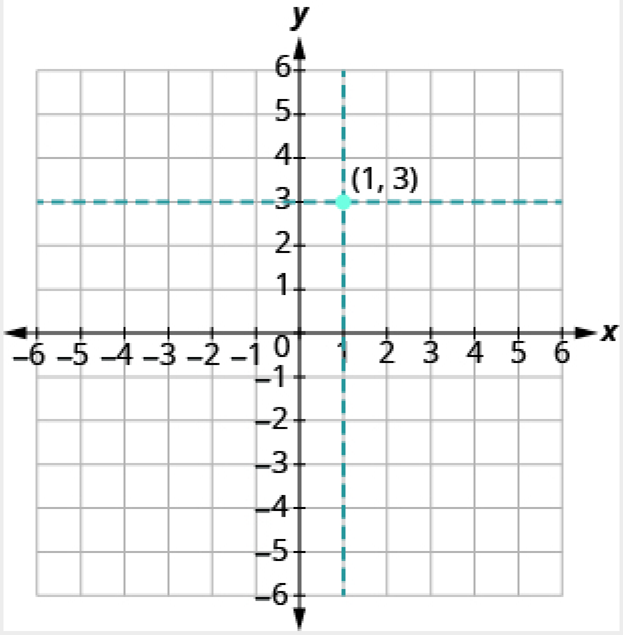
Example:
Exercise:

Problem:

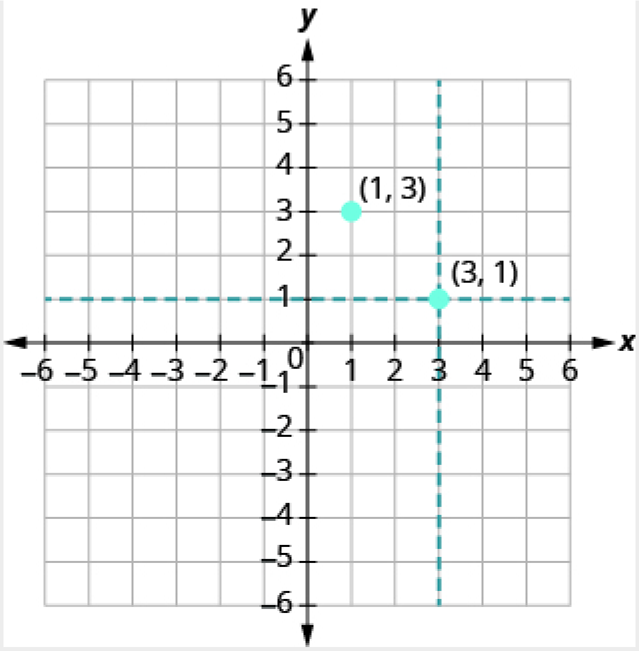
Plot $(1,3)$ and $(3,1)$ in the same rectangular coordinate system.

Solution:
Solution

The coordinate values are the same for both points, but the x and y values are reversed. Let's begin with point $(1,3)$. The x-coordinate is 1 so find 1 on the x-axis and sketch a vertical line through $x=1$. The y-coordinate is 3 so we find 3 on the y-axis and sketch a horizontal line through $y=3$. Where the two lines meet, we plot the point $(1,3)$.



To plot the point $(3,1)$, we start by locating 3 on the x-axis and sketch a vertical line through $x=3$. Then we find 1 on the y-axis and sketch a horizontal line through $y=1$. Where the two lines meet, we plot the point $(3,1)$.



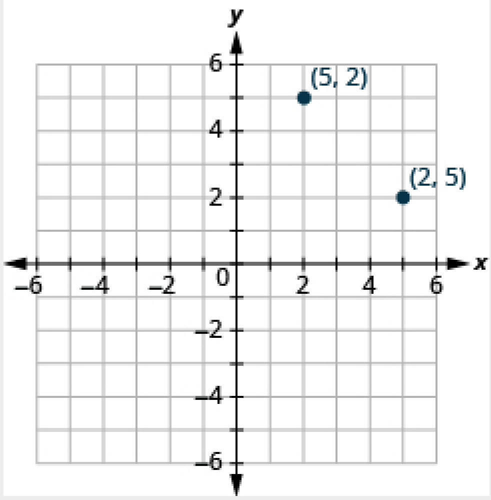
Notice that the order of the coordinates does matter, so, $(1,3)$ is not the same point as $(3,1)$.

Note:
Exercise:

Problem:

Plot each point on the same rectangular coordinate system: $(2,5),(5,2)$.

Solution:

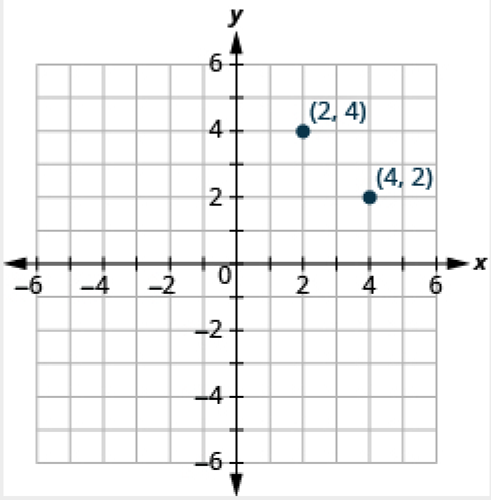


Note:
Exercise:

Problem:

Plot each point on the same rectangular coordinate system: $(4,2),(2,4)$.

Solution:



Example:
Exercise:

Problem:

Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-1, 3)$
ⓑ $(-3, -4)$
ⓒ $(2, -3)$
ⓓ $\left(3, \frac{5}{2}\right)$

Solution:
Solution

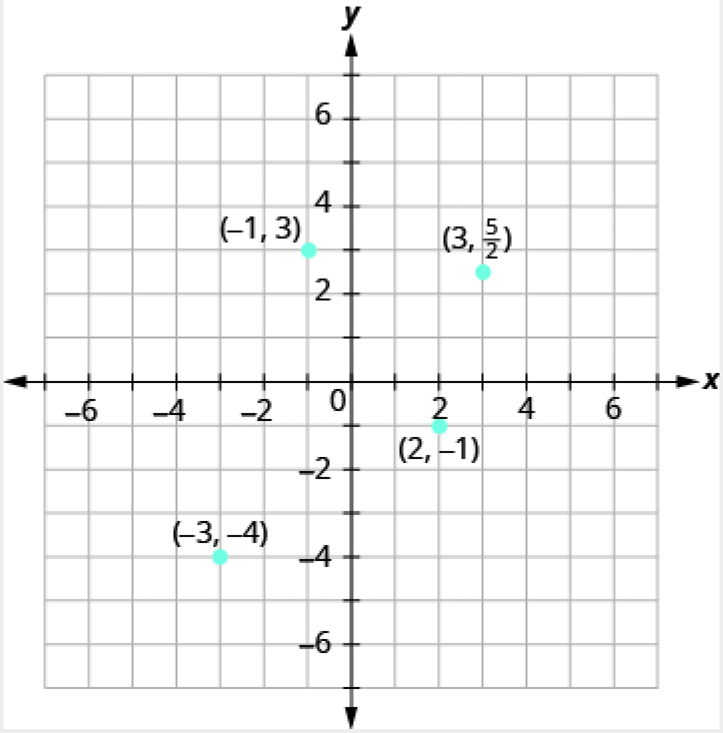
The first number of the coordinate pair is the x-coordinate, and the second number is the y-coordinate.

ⓐ Since $x = -1, y = 3$, the point $(-1, 3)$ is in Quadrant II.

ⓑ Since $x = -3, y = -4$, the point $(-3, -4)$ is in Quadrant III.

ⓒ Since $x = 2, y = -1$, the point $(2, -1)$ is in Quadrant lV.

ⓓ Since $x = 3, y = \frac{5}{2}$, the point $\left(3, \frac{5}{2}\right)$ is in Quadrant I. It may be helpful to write $\frac{5}{2}$ as the mixed number, $2\frac{1}{2}$, or decimal, 2.5. Then we know that the point is halfway between 2 and 3 on the y-axis.



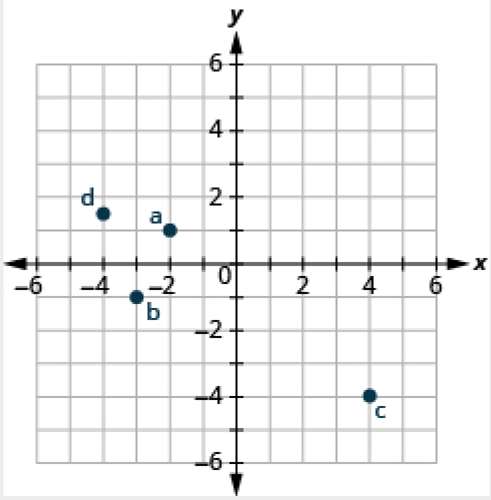
Note:
Exercise:

Problem:

Plot each point on a rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-2, 1)$
ⓑ $(-3, -1)$
ⓒ $(4, -4)$
ⓓ $\left(-4, \frac{3}{2}\right)$

Solution:



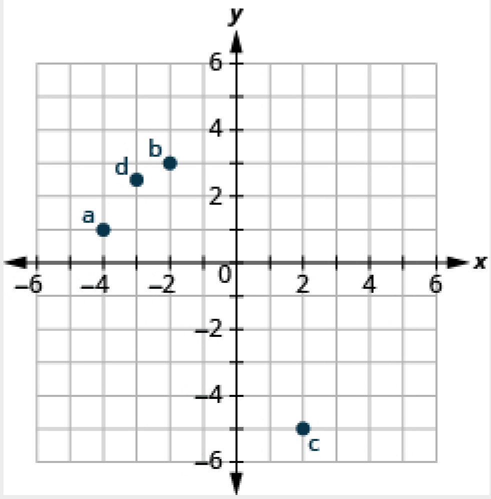
Note:
Exercise:

Problem:

Plot each point on a rectangular coordinate system and identify the quadrant in which the point is located

ⓐ $(-4, 1)$
ⓑ $(-2, 3)$
ⓒ $(2, -5)$
ⓓ $\left(-3, \frac{5}{2}\right)$

Solution:



How do the signs affect the location of the points?

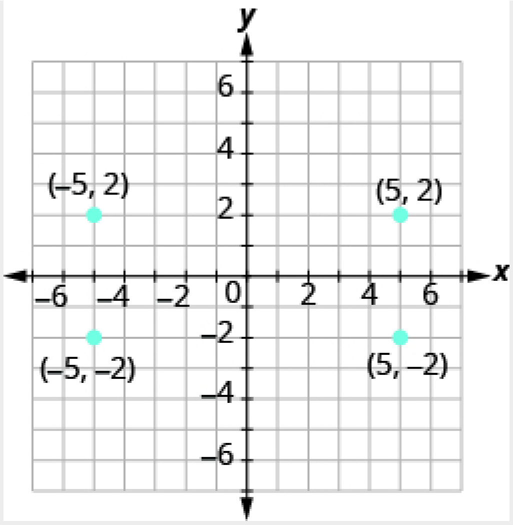
Example:
Exercise:

Problem: Plot each point:

ⓐ $(-5, 2)$
ⓑ $(-5, -2)$
ⓒ $(5, 2)$
ⓓ $(5, -2)$

Solution:
Solution

As we locate the x-coordinate and the y-coordinate, we must be careful with the signs.

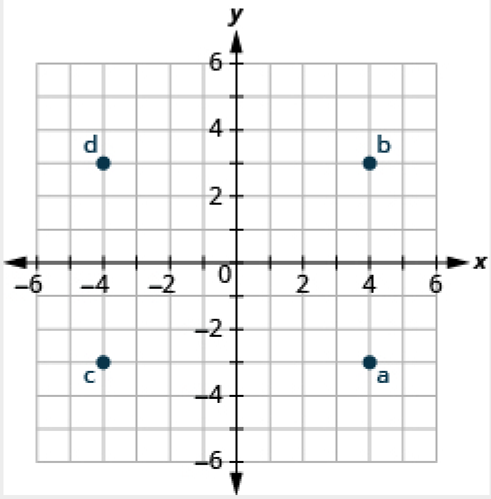


Note:
Exercise:

Problem: Plot each point:

ⓐ $(4, -3)$
ⓑ $(4, 3)$
ⓒ $(-4, -3)$
ⓓ $(-4, 3)$

Solution:

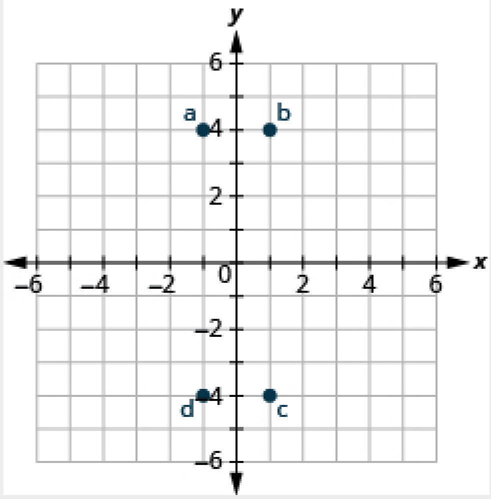


Note:
Exercise:

Problem: Plot each point:

ⓐ $(-1, 4)$
ⓑ $(1, 4)$
ⓒ $(1, -4)$
ⓓ $(-1, -4)$

Solution:

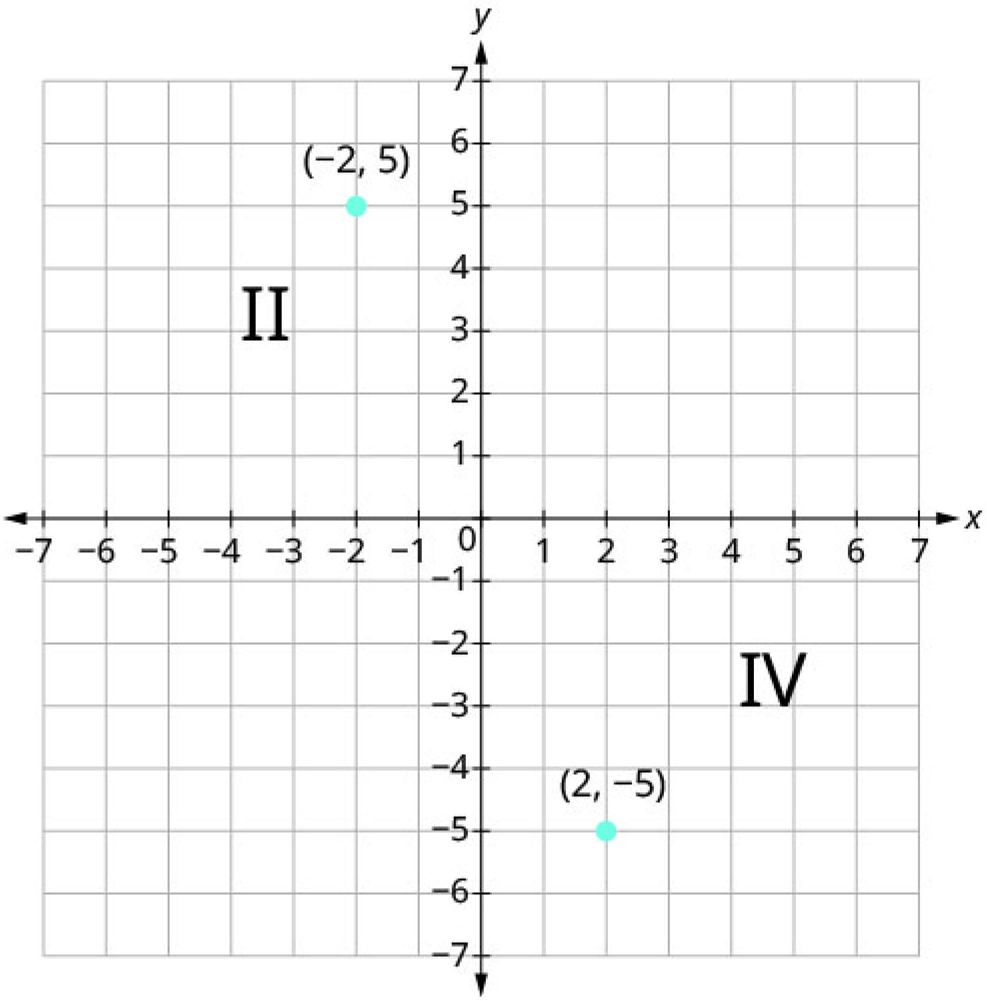


You may have noticed some patterns as you graphed the points in the two previous examples.

For each point in Quadrant IV, what do you notice about the signs of the coordinates?

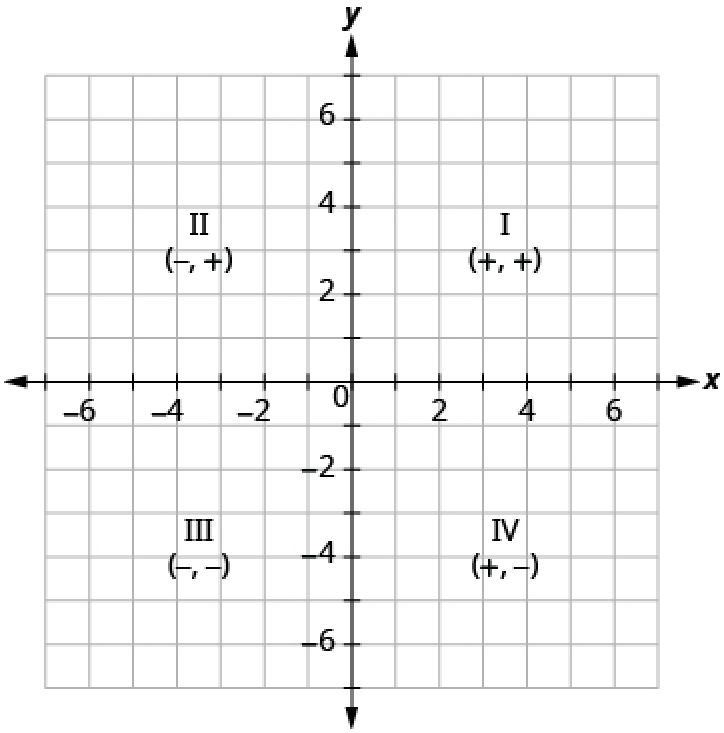
What about the signs of the coordinates of the points in the third quadrant? The second quadrant? The first quadrant?

Can you tell just by looking at the coordinates in which quadrant the point (−2, 5) is located? In which quadrant is (2, −5) located?



We can summarize sign patterns of the quadrants as follows. Also see [link].

Quadrant I	**Quadrant II**	**Quadrant III**	**Quadrant IV**
(x,y)	(x,y)	(x,y)	(x,y)
$(+,+)$	$(−,+)$	$(−,−)$	$(+,−)$



What if one coordinate is zero? Where is the point $(0, 4)$ located? Where is the point $(-2, 0)$ located? The point $(0, 4)$ is on the *y*-axis and the point $(-2, 0)$ is on the *x*-axis.

Note:
Points on the Axes
Points with a y-coordinate equal to 0 are on the x-axis, and have coordinates $(a, 0)$.
Points with an x-coordinate equal to 0 are on the y-axis, and have coordinates $(0, b)$.
What is the ordered pair of the point where the axes cross? At that point both coordinates are zero, so its ordered pair is $(0, 0)$. The point has a special name. It is called the *origin*.

Note:
The Origin

The point $(0,0)$ is called the **origin**. It is the point where the *x*-axis and *y*-axis intersect.

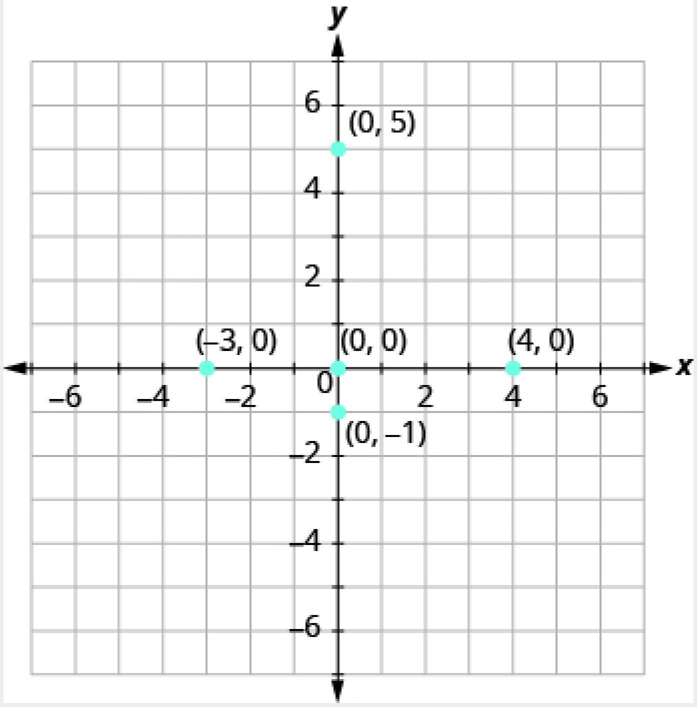
Example:
Exercise:

Problem: Plot each point on a coordinate grid:

ⓐ $(0,5)$
ⓑ $(4,0)$
ⓒ $(-3,0)$
ⓓ $(0,0)$
ⓔ $(0,-1)$

Solution:
Solution

ⓐ Since $x=0$, the point whose coordinates are $(0,5)$ is on the y-axis.
ⓑ Since $y=0$, the point whose coordinates are $(4,0)$ is on the x-axis.
ⓒ Since $y=0$, the point whose coordinates are $(-3,0)$ is on the x-axis.
ⓓ Since $x=0$ and $y=0$, the point whose coordinates are $(0,0)$ is the origin.
ⓔ Since $x=0$, the point whose coordinates are $(0,-1)$ is on the y-axis.

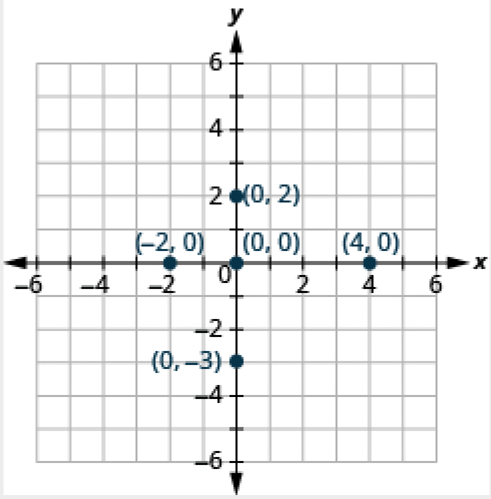


Note:
Exercise:

Problem: Plot each point on a coordinate grid:

ⓐ $(4, 0)$
ⓑ $(-2, 0)$
ⓒ $(0, 0)$
ⓓ $(0, 2)$
ⓔ $(0, -3)$

Solution:

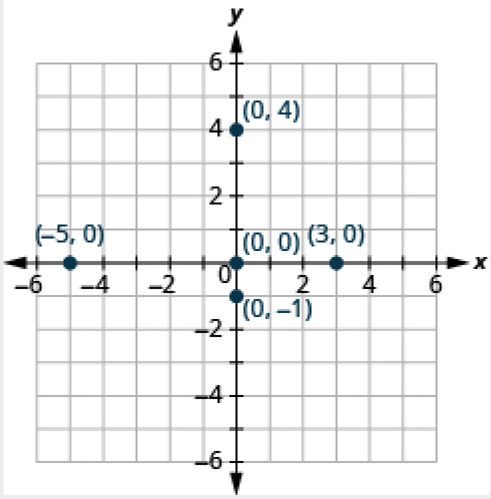


Note:
Exercise:

Problem: Plot each point on a coordinate grid:

ⓐ $(-5,0)$
ⓑ $(3,0)$
ⓒ $(0,0)$
ⓓ $(0,-1)$
ⓔ $(0,4)$

Solution:

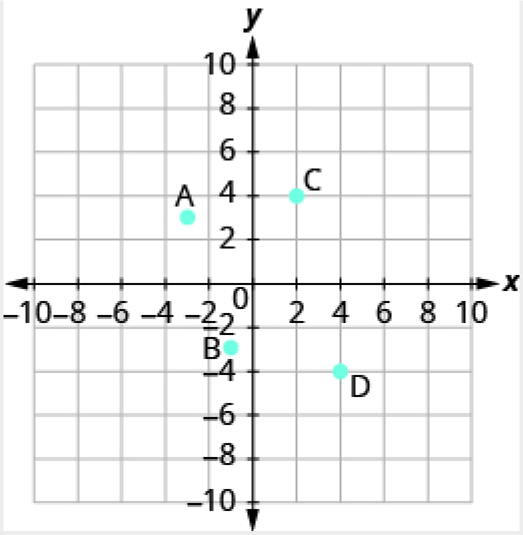


Identify Points on a Graph

In algebra, being able to identify the coordinates of a point shown on a graph is just as important as being able to plot points. To identify the *x*-coordinate of a point on a graph, read the number on the *x*-axis directly above or below the point. To identify the *y*-coordinate of a point, read the number on the *y*-axis directly to the left or right of the point. Remember, to write the ordered pair using the correct order (x, y).

Example:
Exercise:

Problem: Name the ordered pair of each point shown:



Solution:
Solution

Point A is above -3 on the x-axis, so the x-coordinate of the point is -3. The point is to the left of 3 on the y-axis, so the y-coordinate of the point is 3. The coordinates of the point are $(-3, 3)$.

Point B is below -1 on the x-axis, so the x-coordinate of the point is -1. The point is to the left of -3 on the y-axis, so the y-coordinate of the point is -3. The coordinates of the point are $(-1, -3)$.

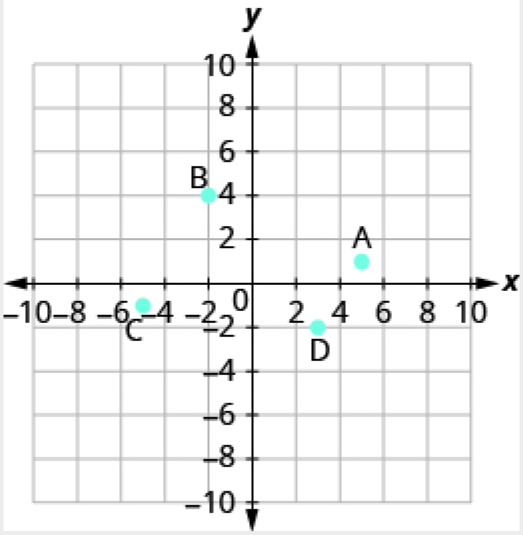
Point C is above 2 on the x-axis, so the x-coordinate of the point is 2. The point is to the right of 4 on the y-axis, so the y-coordinate of the point is 4. The coordinates of the point are $(2, 4)$.

Point D is below 4 on the $x -$ axis, so the x-coordinate of the point is 4. The point is to the right of -4 on the y-axis, so the y-coordinate of the point is -4. The coordinates of the point are $(4, -4)$.

Note:

Exercise:

Problem: Name the ordered pair of each point shown:

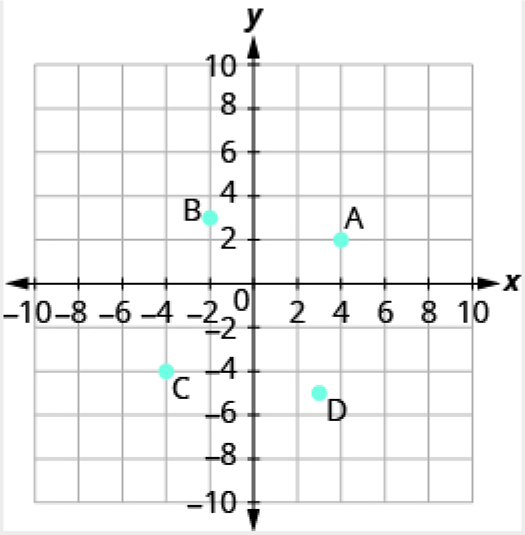


Solution:

1. A: (5,1)
2. B: (−2,4)
3. C: (−5,−1)
4. D: (3,−2)

Note:
Exercise:

Problem: Name the ordered pair of each point shown:

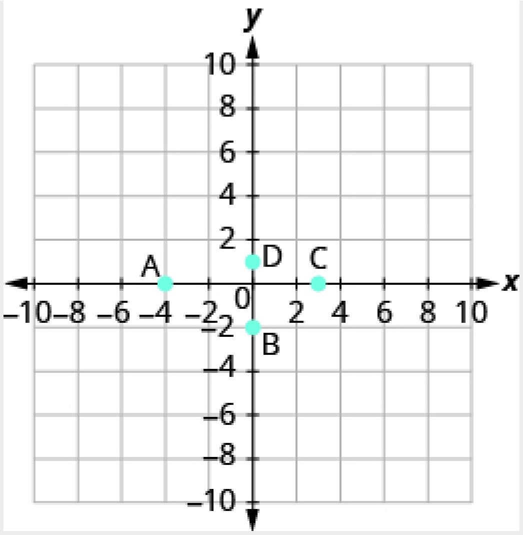


Solution:

1. A: (4,2)
2. B: (−2,3)
3. C: (−4,−4)
4. D: (3,−5)

Example:
Exercise:

Problem: Name the ordered pair of each point shown:

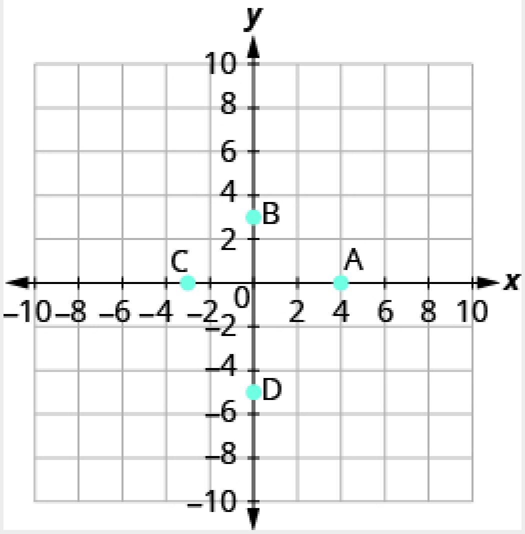


Solution:
Solution

Point A is on the *x*-axis at $x = -4$.	The coordinates of point A are $(-4, 0)$.
Point B is on the *y*-axis at $y = -2$	The coordinates of point B are $(0, -2)$.
Point C is on the *x*-axis at $x = 3$.	The coordinates of point C are $(3, 0)$.
Point D is on the *y*-axis at $y = 1$.	The coordinates of point D are $(0, 1)$.

Note:
Exercise:

Problem: Name the ordered pair of each point shown:

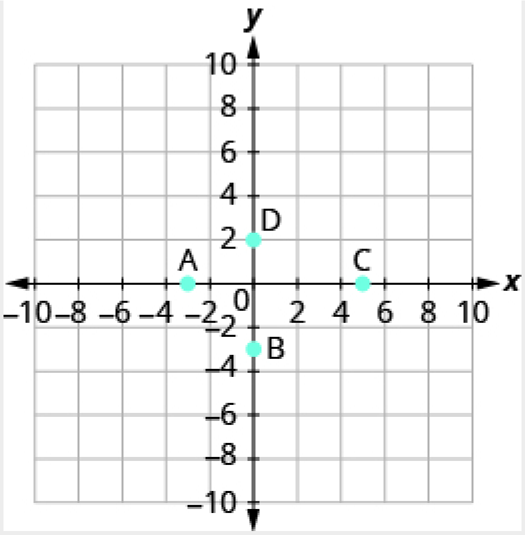


Solution:

1. A: (4,0)
2. B: (0,3)
3. C: (−3,0)
4. D: (0,−5)

Note:
Exercise:

Problem: Name the ordered pair of each point shown:



Solution:

1. A: (−3,0)
2. B: (0,−3)
3. C: (5,0)
4. D: (0,2)

Verify Solutions to an Equation in Two Variables

All the equations we solved so far have been equations with one variable. In almost every case, when we solved the equation we got exactly one solution. The process of solving an equation ended with a statement such as $x = 4$. Then we checked the solution by substituting back into the equation.

Here's an example of a linear equation in one variable, and its one solution.
Equation:

$$3x + 5 = 17$$

$$3x = 12$$

$$x = 4$$

But equations can have more than one variable. Equations with two variables can be written in the general form $Ax + By = C$. An equation of this form is called a linear equation in two variables.

> **Note:**
> Linear Equation
> An equation of the form $Ax + By = C$, where A and B are not both zero, is called a linear equation in two variables.

Notice that the word “line” is in linear.

Here is an example of a linear equation in two variables, x and y:

$$Ax + By = C$$

$$x + 4y = 8$$

$$A = 1, B = 4, C = 8$$

Is $y = -5x + 1$ a linear equation? It does not appear to be in the form $Ax + By = C$. But we could rewrite it in this form.

	$y = -5x + 1$
Add $5x$ to both sides.	$y + 5x = -5x + 1 + 5x$

Simplify.	$y + 5x = 1$
Use the Commutative Property to put it in $Ax + By = C$.	$Ax + By = C$ $5x + y = 1$

By rewriting $y = -5x + 1$ as $5x + y = 1$, we can see that it is a linear equation in two variables because it can be written in the form $Ax + By = C$.

Linear equations in two variables have infinitely many solutions. For every number that is substituted for x, there is a corresponding y value. This pair of values is a **solution to the linear equation** and is represented by the ordered pair (x, y). When we substitute these values of x and y into the equation, the result is a true statement because the value on the left side is equal to the value on the right side.

Note:
Solution to a Linear Equation in Two Variables
An ordered pair (x, y) is a solution to the linear equation $Ax + By = C$, if the equation is a true statement when the x- and y-values of the ordered pair are substituted into the equation.

Example:
Exercise:

Problem:

Determine which ordered pairs are solutions of the equation $x + 4y = 8$:

ⓐ $(0, 2)$
ⓑ $(2, -4)$

ⓒ $(-4, 3)$

Solution:
Solution

Substitute the x- and y-values from each ordered pair into the equation and determine if the result is a true statement.

ⓐ $(0, 2)$	ⓑ $(2, -4)$	ⓒ $(-4, 3)$
$x = 0, y = 2$ $x + 4y = 8$ $0 + 4 \cdot 2 \stackrel{?}{=} 8$ $0 + 8 \stackrel{?}{=} 8$ $8 = 8$ ✓	$x = 2, y = -4$ $x + 4y = 8$ $2 + 4(-4) \stackrel{?}{=} 8$ $2 + (-16) \stackrel{?}{=} 8$ $-14 \neq 8$	$x = -4, y = 3$ $x + 4y = 8$ $-4 + 4 \cdot 3 \stackrel{?}{=} 8$ $-4 + 12 \stackrel{?}{=} 8$ $8 = 8$ ✓
$(0, 2)$ is a solution.	$(2, -4)$ is not a solution.	$(-4, 3)$ is a solution.

Note:
Exercise:

Problem:

Determine which ordered pairs are solutions to the given equation: $2x + 3y = 6$

ⓐ $(3, 0)$
ⓑ $(2, 0)$
ⓒ $(6, -2)$

Solution:

ⓐ, ⓒ

Note:
Exercise:

Problem:

Determine which ordered pairs are solutions to the given equation: $4x - y = 8$

ⓐ $(0, 8)$
ⓑ $(2, 0)$
ⓒ $(1, -4)$

Solution:

ⓑ, ⓒ

Example:
Exercise:

Problem:

Determine which ordered pairs are solutions of the equation.
$y = 5x - 1$:

ⓐ $(0, -1)$
ⓑ $(1, 4)$
ⓒ $(-2, -7)$

Solution:
Solution

Substitute the x- and y-values from each ordered pair into the equation and determine if it results in a true statement.

ⓐ $(0, -1)$	ⓑ $(1, 4)$	ⓒ $(-2, -7)$
$x = 0, y = -1$ $y = 5x - 1$ $-1 \stackrel{?}{=} 5(0) - 1$ $-1 \stackrel{?}{=} 0 - 1$ $-1 = -1$ ✓	$x = 1, y = 4$ $y = 5x - 1$ $4 \stackrel{?}{=} 5(1) - 1$ $4 \stackrel{?}{=} 5 - 1$ $4 = 4$ ✓	$x = -2, y = -7$ $y = 5x - 1$ $-7 \stackrel{?}{=} 5(-2) - 1$ $-7 \stackrel{?}{=} -10 - 1$ $-7 \neq -11$
$(0, -1)$ is a solution.	$(1, 4)$ is a solution.	$(-2, -7)$ is not a solution.

Note:
Exercise:

Problem:

Determine which ordered pairs are solutions of the given equation: $y = 4x - 3$

ⓐ $(0, 3)$
ⓑ $(1, 1)$
ⓒ $(1, 1)$

Solution:

ⓑ

Note:
Exercise:

Problem:

Determine which ordered pairs are solutions of the given equation: $y = -2x + 6$

ⓐ $(0, 6)$
ⓑ $(1, 4)$
ⓒ $(-2, -2)$

Solution:

ⓐ, ⓑ

Complete a Table of Solutions to a Linear Equation

In the previous examples, we substituted the x- and y-values of a given ordered pair to determine whether or not it was a solution to a linear equation. But how do we find the ordered pairs if they are not given? One way is to choose a value for x and then solve the equation for y. Or, choose a value for y and then solve for x.

We’ll start by looking at the solutions to the equation $y=5x-1$ we found in [link]. We can summarize this information in a table of solutions.

$y=5x-1$		
x	y	(x,y)
0	-1	$(0,-1)$
1	4	$(1,4)$

To find a third solution, we’ll let $x=2$ and solve for y.

	$y=5x-1$
Substitute $x = 2$.	$y=5(2)-1$

Multiply.	$y = 10 - 1$
Simplify.	$y = 9$

The ordered pair is a solution to $y = 5x - 1$. We will add it to the table.

$y = 5x - 1$		
x	y	(x, y)
0	-1	$(0, -1)$
1	4	$(1, 4)$
2	9	$(2, 9)$

We can find more solutions to the equation by substituting any value of x or any value of y and solving the resulting equation to get another ordered pair that is a solution. There are an infinite number of solutions for this equation.

Example:
Exercise:

Problem:

Complete the table to find three solutions to the equation $y = 4x - 2$:

$y = 4x - 2$		
x	y	(x, y)
0		
-1		
2		

Solution:
Solution

Substitute $x = 0$, $x = -1$, and $x = 2$ into $y = 4x - 2$.

$x = 0$	$x = -1$	$x = 2$
$y = 4x - 2$	$y = 4x - 2$	$y = 4x - 2$
$y = 4 \cdot 0 - 2$	$y = 4(-1) - 2$	$y = 4 \cdot 2 - 2$
$y = 0 - 2$	$y = -4 - 2$	$y = 8 - 2$
$y = -2$	$y = -6$	$y = 6$
$(0, -2)$	$(-1, -6)$	$(2, 6)$

The results are summarized in the table.

$y = 4x - 2$		
x	y	(x, y)
0	-2	$(0, -2)$
-1	-6	$(-1, -6)$
2	6	$(2, 6)$

Note:
Exercise:

Problem:

Complete the table to find three solutions to the equation: $y = 3x - 1.$

$y = 3x - 1$		
x	y	(x, y)

$y = 3x - 1$		
x	y	(x, y)
0		
-1		
2		

Solution:

$y = 3x - 1$		
x	y	(x, y)
0	-1	$(0, -1)$
-1	-4	$(-1, -4)$
2	5	$(2, 5)$

Note:
Exercise:

Problem:

Complete the table to find three solutions to the equation: $y = 6x + 1$

$y = 6x + 1$		
x	y	(x, y)
0		
1		
-2		

Solution:

$y = 6x + 1$		
x	y	(x, y)
0	1	$(0, 1)$
1	7	$(1, 7)$
-2	-11	$(-2, -11)$

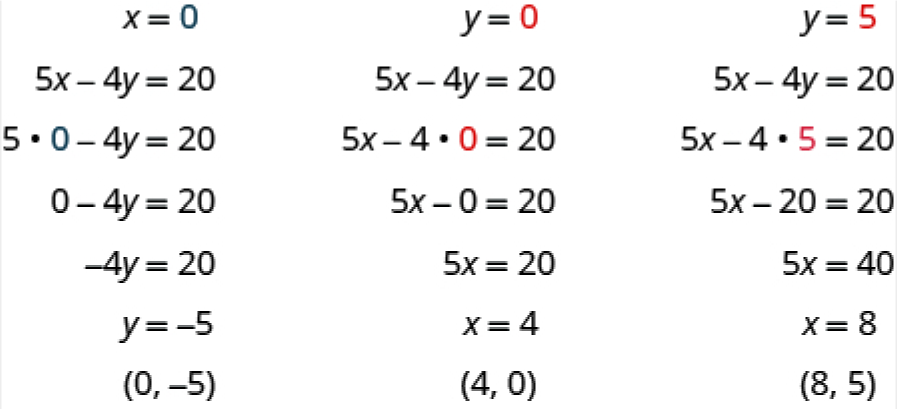
Example:
Exercise:

Problem:

Complete the table to find three solutions to the equation $5x - 4y = 20$:

$5x - 4y = 20$		
x	y	(x, y)
0		
	0	
	5	

Solution:
Solution



$x = 0$	$y = 0$	$y = 5$
$5x - 4y = 20$	$5x - 4y = 20$	$5x - 4y = 20$
$5 \cdot 0 - 4y = 20$	$5x - 4 \cdot 0 = 20$	$5x - 4 \cdot 5 = 20$
$0 - 4y = 20$	$5x - 0 = 20$	$5x - 20 = 20$
$-4y = 20$	$5x = 20$	$5x = 40$
$y = -5$	$x = 4$	$x = 8$
$(0, -5)$	$(4, 0)$	$(8, 5)$

The results are summarized in the table.

$5x - 4y = 20$		
x	y	(x, y)
0	-5	$(0, -5)$
4	0	$(4, 0)$
8	5	$(8, 5)$

Note:

Exercise:

Problem:

Complete the table to find three solutions to the equation: $2x - 5y = 20$.

$2x-5y=20$		
x	y	(x,y)
0		
	0	
-5		

Solution:

$2x-5y=20$		
x	y	(x,y)
0	-4	$(0,-4)$
10	0	$(10,0)$
-5	-6	$(-5,-6)$

Note:
Exercise:

Problem:

Complete the table to find three solutions to the equation: $3x - 4y = 12$.

$3x - 4y = 12$		
x	y	(x, y)
0		
	0	
-4		

Solution:

$3x - 4y = 12$		
x	y	(x, y)
0	-3	$(0, -3)$
4	0	$(4, 0)$

$3x - 4y = 12$		
x	y	(x, y)
-4	-6	$(-4, -6)$

Find Solutions to Linear Equations in Two Variables

To find a solution to a linear equation, we can choose any number we want to substitute into the equation for either x or y. We could choose $1, 100, 1{,}000$, or any other value we want. But it's a good idea to choose a number that's easy to work with. We'll usually choose 0 as one of our values.

Example:
Exercise:

Problem: Find a solution to the equation $3x + 2y = 6$.

Solution:
Solution

Step 1: Choose any value for one of the variables in the equation.		We can substitute any value we want for x or any

		value for y. Let's pick $x = 0$. What is the value of y if $x = 0$?
Step 2: Substitute that value into the equation. Solve for the other variable.	Substitute 0 for x. Simplify. Divide both sides by 2.	$3x + 2y = 6$ $3 \cdot 0 + 2y = 6$ $0 + 2y = 6$ $2y = 6$ $y = 3$
Step 3: Write the solution as an ordered pair.	So, when $x = 0, y = 3$.	This solution is represented by the ordered pair $(0, 3)$.
Step 4: Check.	Substitute $x = 0, y = 3$ into the equation $3x + 2y = 6$. Is the result a true equation? Yes!	$3x + 2y = 6$ $3 \cdot 0 + 2 \cdot 3 \overset{?}{=} 6$ $0 + 6 \overset{?}{=} 6$ $6 = 6$ ✓

Note:
Exercise:

Problem: Find a solution to the equation: $4x + 3y = 12$.

Solution:

Answers will vary.

Note:
Exercise:

Problem: Find a solution to the equation: $2x + 4y = 8$.

Solution:

Answers will vary.

We said that linear equations in two variables have infinitely many solutions, and we've just found one of them. Let's find some other solutions to the equation $3x + 2y = 6$.

Example:
Exercise:

Problem: Find three more solutions to the equation $3x + 2y = 6$.

Solution:
Solution

To find solutions to $3x + 2y = 6$, choose a value for x or y. Remember, we can choose any value we want for x or y. Here we chose 1 for x, and 0 and -3 for y.

Substitute it into the equation.	$y = 0$ $3x + 2y = 6$ $3x + 2(0) = 6$	$x = 1$ $3x + 2y = 6$ $3(1) + 2y = 6$	$y = -3$ $3x + 2y = 6$ $3x + 2(-3) = 6$
Simplify. Solve.	$3x + 0 = 6$ $3x = 6$	$3 + 2y = 6$ $2y = 3$	$3x - 6 = 6$ $3x = 12$
	$x = 2$	$y = \frac{3}{2}$	$x = 4$
Write the ordered pair.	$(2, 0)$	$(1, \frac{3}{2})$	$(4, -3)$

Check your answers.

$(2, 0)$	$(1, \frac{3}{2})$	$(4, -3)$
$3x + 2y = 6$ $3 \cdot 2 + 2 \cdot 0 \stackrel{?}{=} 6$ $6 + 0 \stackrel{?}{=} 6$ $6 = 6\checkmark$	$3x + 2y = 6$ $3 \cdot 1 + 2 \cdot \frac{3}{2} \stackrel{?}{=} 6$ $3 + 3 \stackrel{?}{=} 6$ $6 = 6\checkmark$	$3x + 2y = 6$ $3 \cdot 4 + 2(-3) \stackrel{?}{=} 6$ $12 + (-6) \stackrel{?}{=} 6$ $6 = 6\checkmark$

So $(2,0)$, $\left(1,\frac{3}{2}\right)$ and $(4,-3)$ are all solutions to the equation $3x+2y=6$. In the previous example, we found that $(0,3)$ is a solution, too. We can list these solutions in a table.

$3x+2y=6$		
x	y	(x,y)
0	3	$(0,3)$
2	0	$(2,0)$
1	$\frac{3}{2}$	$(1,\frac{3}{2})$
4	-3	$(4,-3)$

Note:
Exercise:

Problem: Find three solutions to the equation: $2x+3y=6$.

Solution:

Answers will vary.

Note:
Exercise:

Problem: Find three solutions to the equation: $4x + 2y = 8$.

Solution:

Answers will vary.

Let's find some solutions to another equation now.

Example:
Exercise:

Problem: Find three solutions to the equation $x - 4y = 8$.

Solution:
Solution

	$x - 4y = 8$	$x - 4y = 8$	$x - 4y = 8$
Choose a value for x or y.	$x = 0$	$y = 0$	$y = 3$
Substitute it into the equation.	$0 - 4y = 8$	$x - 4 \cdot 0 = 8$	$x - 4 \cdot 3 = 8$

Solve.	$-4y=8$ $y=-2$	$x-0=8$ $x=8$	$x-12=8$ $x=20$
Write the ordered pair.	$(0,-2)$	$(8,0)$	$(20,3)$

So $(0,-2)$, $(8,0)$, and $(20,3)$ are three solutions to the equation $x-4y=8$.

$x-4y=8$		
x	y	(x,y)
0	-2	$(0,-2)$
8	0	$(8,0)$
20	3	$(20,3)$

Remember, there are an infinite number of solutions to each linear equation. Any point you find is a solution if it makes the equation true.

Note:
Exercise:

Problem: Find three solutions to the equation: $4x + y = 8$.

Solution:

Answers will vary.

Note:
Exercise:

Problem: Find three solutions to the equation: $x + 5y = 10$.

Solution:

Answers will vary.

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Plotting Points
- Identifying Quadrants
- Verifying Solution to Linear Equation

Key Concepts

- **Sign Patterns of the Quadrants**

Quadrant I	**Quadrant II**	**Quadrant III**	**Quadrant IV**
(x,y)	(x,y)	(x,y)	(x,y)
$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$

- **Coordinates of Zero**
    - Points with a y-coordinate equal to 0 are on the x-axis, and have coordinates $(a, 0)$.
    - Points with a x-coordinate equal to 0 are on the y-axis, and have coordinates $(0, b)$.
    - The point (0, 0) is called the origin. It is the point where the x-axis and y-axis intersect.

Practice Makes Perfect

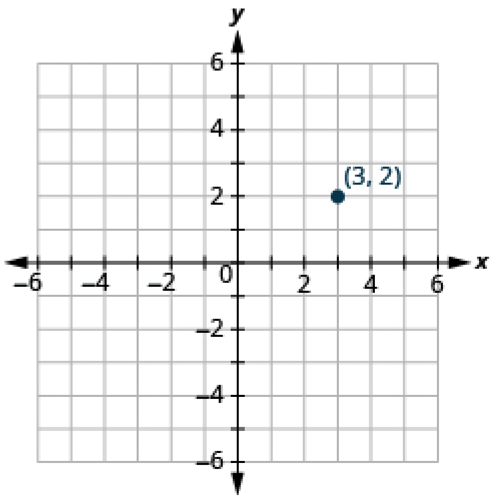
Plot Points on a Rectangular Coordinate System

In the following exercises, plot each point on a coordinate grid.

Exercise:

Problem: $(3, 2)$

Solution:



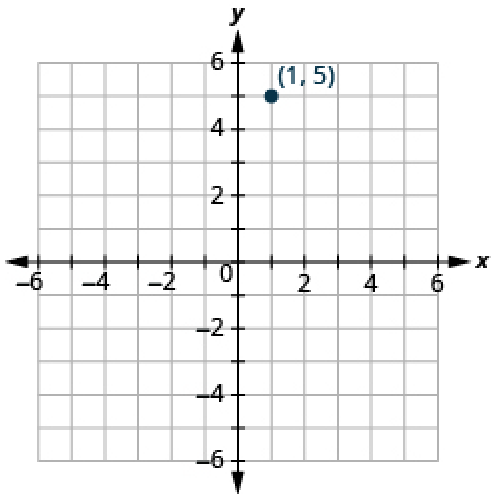
Exercise:

Problem: $(4, 1)$

Exercise:

Problem: $(1, 5)$

Solution:



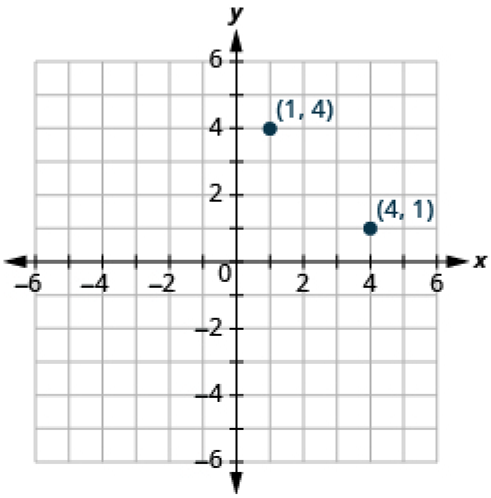
Exercise:

Problem: $(3, 4)$

Exercise:

Problem: $(4, 1), (1, 4)$

Solution:



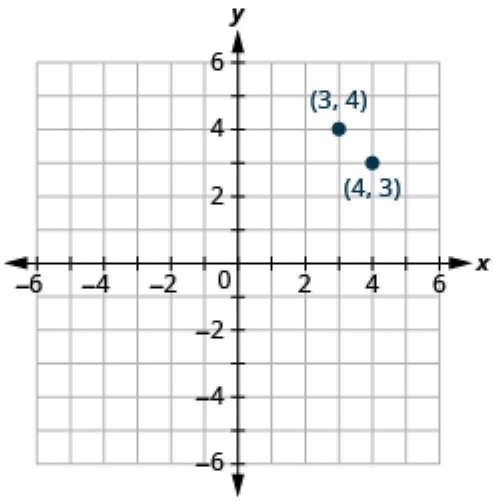
Exercise:

Problem: $(3, 2), (2, 3)$

Exercise:

Problem: $(3, 4), (4, 3)$

Solution:



In the following exercises, plot each point on a coordinate grid and identify the quadrant in which the point is located.

Exercise:

Problem:

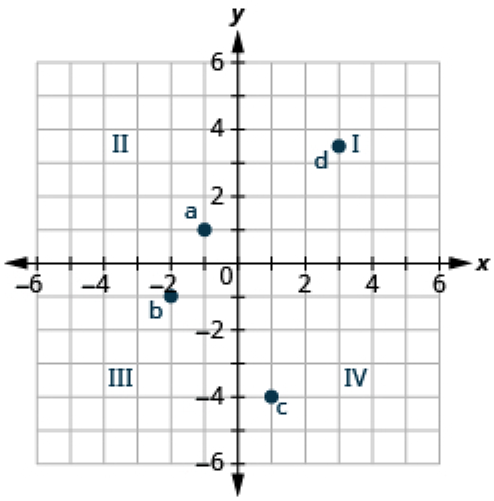
ⓐ $(-4, 2)$
ⓑ $(-1, -2)$
ⓒ $(3, -5)$
ⓓ $\left(2, \frac{5}{2}\right)$

Exercise:

Problem:

ⓐ $(-2, -3)$
ⓑ $(3, -3)$
ⓒ $(-4, 1)$
ⓓ $\left(1, \frac{3}{2}\right)$

Solution:



Exercise:

Problem:

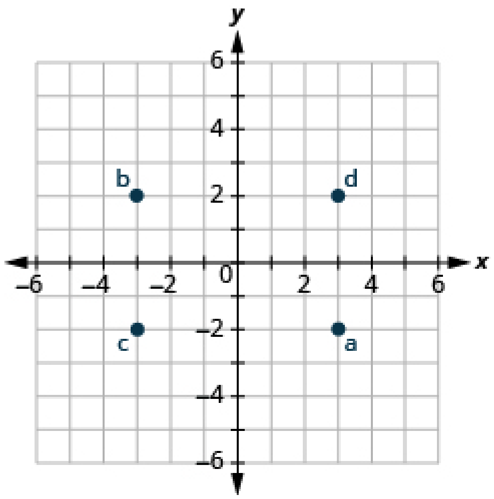
ⓐ $(-1, 1)$
ⓑ $(-2, -1)$
ⓒ $(1, -4)$
ⓓ $\left(3, \frac{7}{2}\right)$

Exercise:

Problem:

ⓐ $(3, -2)$
ⓑ $(-3, 2)$
ⓒ $(-3, -2)$
ⓓ $(3, 2)$

Solution:



Exercise:

Problem:

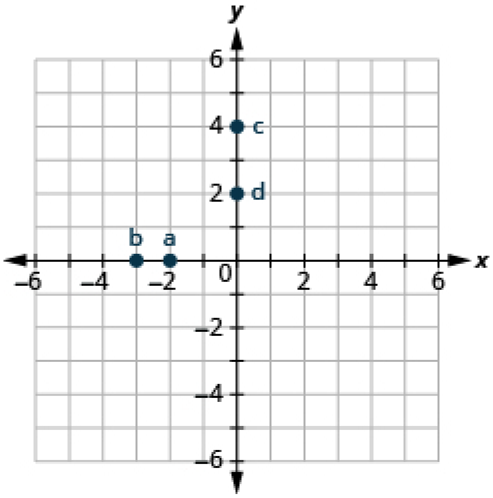
ⓐ $(4, -1)$
ⓑ $(-4, 1)$
ⓒ $(-4, -1)$
ⓓ $(4, 1)$

Exercise:

Problem:

- ⓐ $(-2, 0)$
- ⓑ $(-3, 0)$
- ⓒ $(0, 4)$
- ⓓ $(0, 2)$

Solution:

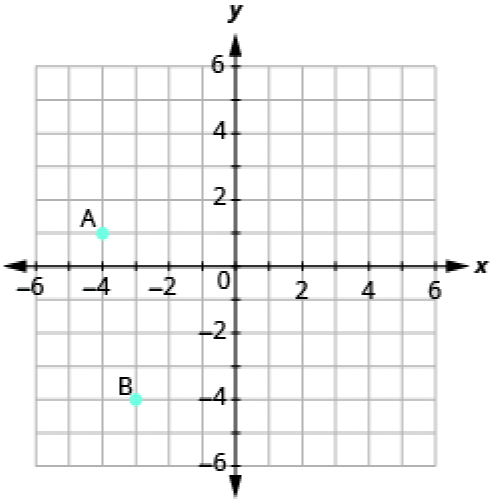


Identify Points on a Graph

In the following exercises, name the ordered pair of each point shown.

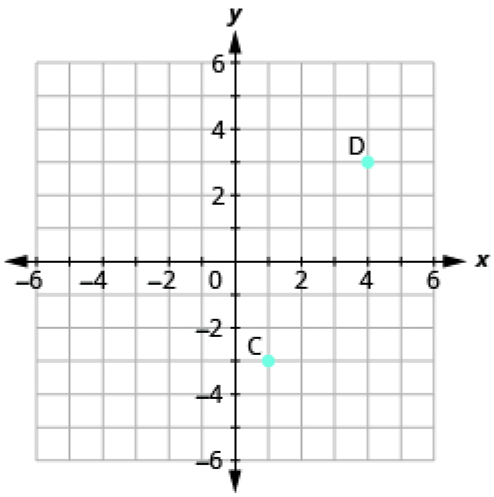
Exercise:

Problem:



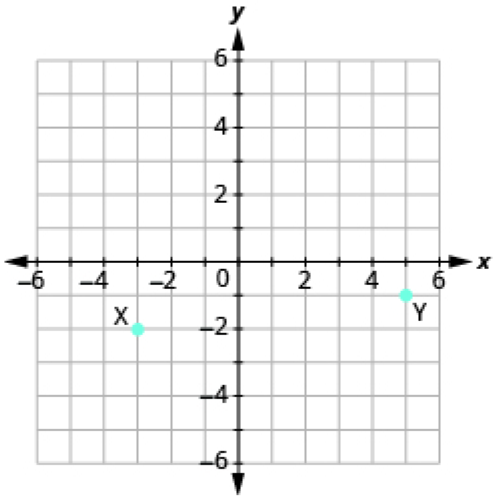
Exercise:

Problem:



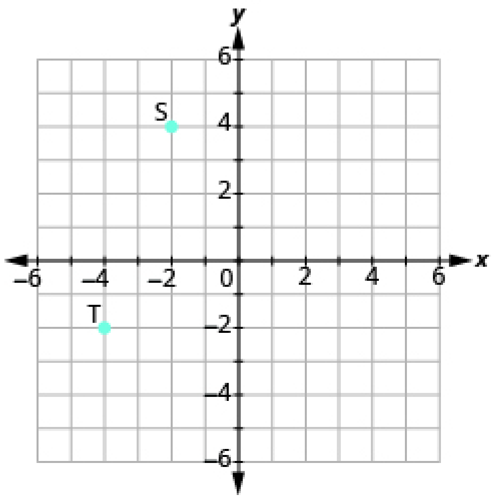
Exercise:

Problem:



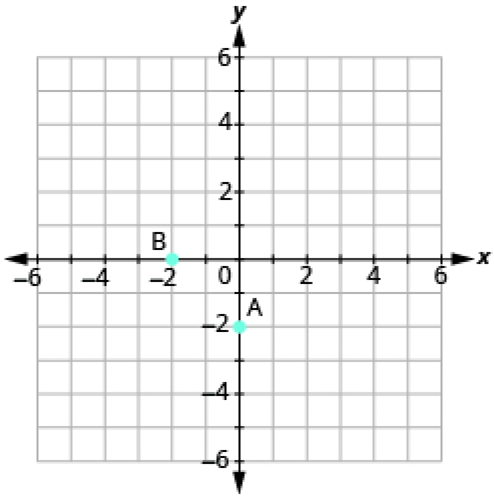
Exercise:

Problem:



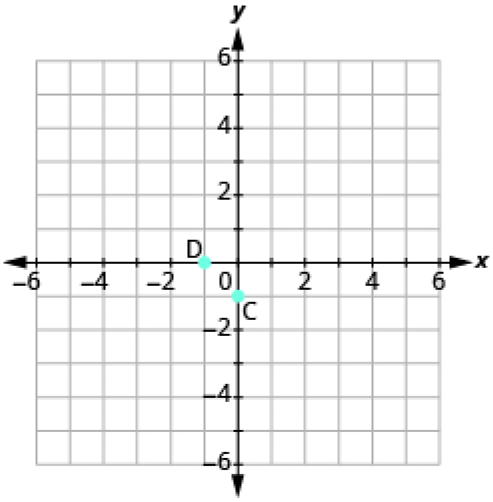
Exercise:

Problem:



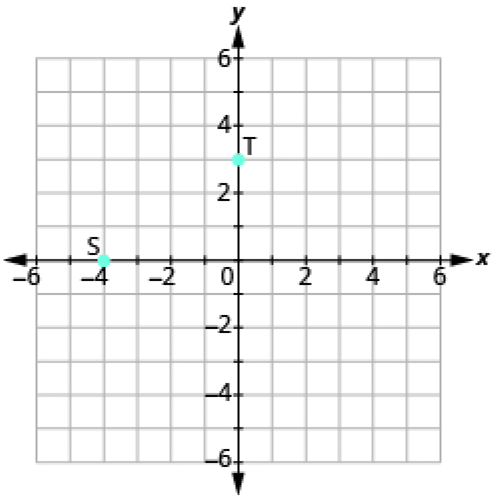
Exercise:

Problem:



Exercise:

Problem:



Verify Solutions to an Equation in Two Variables

In the following exercises, determine which ordered pairs are solutions to the given equation.

Exercise:

Problem: $2x + y = 6$

ⓐ $(1,4)$
ⓑ $(3,0)$
ⓒ $(2,3)$

Solution:

ⓐ, ⓑ

Exercise:

Problem: $x+3y=9$

ⓐ $(0,3)$
ⓑ $(6,1)$
ⓒ $(-3,-3)$

Exercise:

Problem: $4x-2y=8$

ⓐ $(3,2)$
ⓑ $(1,4)$
ⓒ $(0,-4)$

Solution:

ⓐ, ⓒ

Exercise:

Problem: $3x-2y=12$

ⓐ $(4,0)$
ⓑ $(2,-3)$
ⓒ $(1,6)$

Exercise:

Problem: $y = 4x + 3$

ⓐ $(4, 3)$
ⓑ $(-1, -1)$
ⓒ $(\frac{1}{2}, 5)$

Solution:

ⓑ, ⓒ

Exercise:

Problem: $y = 2x - 5$

ⓐ $(0, -5)$
ⓑ $(2, 1)$
ⓒ $(\frac{1}{2}, -4)$

Exercise:

Problem: $y = \frac{1}{2}x - 1$

ⓐ $(2, 0)$
ⓑ $(-6, -4)$
ⓒ $(-4, -1)$

Solution:

ⓐ, ⓑ

Exercise:

Problem: $y = \frac{1}{3}x + 1$

ⓐ $(-3, 0)$
ⓑ $(9, 4)$
ⓒ $(-6, -1)$

Find Solutions to Linear Equations in Two Variables

In the following exercises, complete the table to find solutions to each linear equation.

Exercise:

Problem: $y = 2x - 4$

x	y	(x, y)
-1		
0		
2		

Solution:

x	y	(x, y)
-1	-6	$(-1, -6)$
0	-4	$(0, -4)$
2	0	$(2, 0)$

Exercise:

Problem: $y = 3x - 1$

x	y	(x, y)
-1		
0		
2		

Exercise:

Problem: $y = -x + 5$

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{(x, y)}$
-2		
0		
3		

Solution:

x	y	(x, y)
-2	7	$(-2, 7)$
0	5	$(0, 5)$
3	2	$(3, 2)$

Exercise:

Problem: $y = \frac{1}{3}x + 1$

x	y	(x, y)

x	y	(x, y)
0		
3		
6		

Exercise:

Problem: $y = -\frac{3}{2}x - 2$

x	y	(x, y)
-2		
0		
2		

Solution:

x	y	(x, y)

x	y	(x, y)
-2	1	$(-2, 1)$
0	-2	$(0, -2)$
2	-5	$(2, -5)$

Exercise:

Problem: $x + 2y = 8$

x	y	(x, y)
0		
4		
	0	

Everyday Math

Exercise:

Problem:

Weight of a baby Mackenzie recorded her baby's weight every two months. The baby's age, in months, and weight, in pounds, are listed in the table, and shown as an ordered pair in the third column.

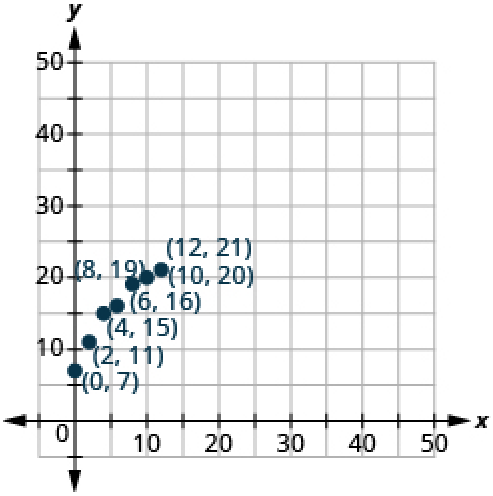
ⓐ Plot the points on a coordinate grid.

Age	Weight	(x, y)
0	7	$(0, 7)$
2	11	$(2, 11)$
4	15	$(4, 15)$
6	16	$(6, 16)$
8	19	$(8, 19)$
10	20	$(10, 20)$
12	21	$(12, 21)$

ⓑ Why is only Quadrant I needed?

Solution:

ⓐ



ⓑ Age and weight are only positive.

Exercise:

Problem:

Weight of a child Latresha recorded her son’s height and weight every year. His height, in inches, and weight, in pounds, are listed in the table, and shown as an ordered pair in the third column.

ⓐ Plot the points on a coordinate grid.

Height x	Weight y	(x, y)
28	22	$(28, 22)$
31	27	$(31, 27)$
33	33	$(33, 33)$

37	35	$(37, 35)$
40	41	$(40, 41)$
42	45	$(42, 45)$

ⓑ Why is only Quadrant I needed?

Writing Exercises

Exercise:

Problem:

Have you ever used a map with a rectangular coordinate system? Describe the map and how you used it.

Solution:

Answers may vary.

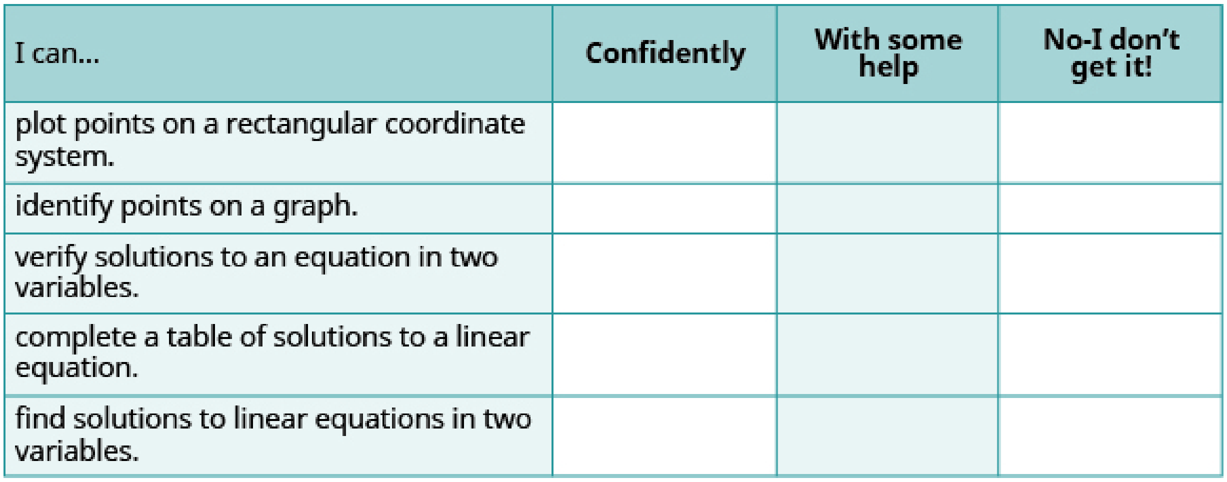
Exercise:

Problem:

How do you determine if an ordered pair is a solution to a given equation?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
plot points on a rectangular coordinate system.			
identify points on a graph.			
verify solutions to an equation in two variables.			
complete a table of solutions to a linear equation.			
find solutions to linear equations in two variables.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

linear equation

An equation of the form $Ax+By=C$, where A and B are not both zero, is called a linear equation in two variables.

ordered pair

An ordered pair (x, y) gives the coordinates of a point in a rectangular coordinate system. The first number is the x-coordinate. The second number is the y-coordinate.

Equation:

$$\underset{x\text{-coordinate},y\text{-coordinate}}{(x, y)}$$

origin

The point $(0, 0)$ is called the origin. It is the point where the the point where the x-axis and y-axis intersect.

quadrants

The x-axis and y-axis divide a rectangular coordinate system into four areas, called quadrants.

solution to a linear equation in two variables

An ordered pair (x, y) is a solution to the linear equation $Ax + By = C$, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

x-axis

The *x*-axis is the horizontal axis in a rectangular coordinate system.

y-axis

The *y*-axis is the vertical axis on a rectangular coordinate system.

Understand Percent

By the end of this section, you will be able to:

- Use the definition of percent
- Convert percents to fractions and decimals
- Convert decimals and fractions to percents

Note:
Before you get started, take this readiness quiz.

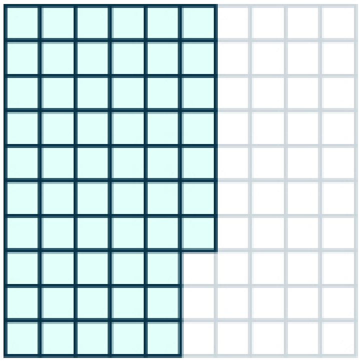
1. Translate “the ratio of 33 to 5” into an algebraic expression.
   If you missed this problem, review [link].
2. Write $\frac{3}{5}$ as a decimal.
   If you missed this problem, review [link].
3. Write 0.62 as a fraction.
   If you missed this problem, review [link].

Use the Definition of Percent

How many cents are in one dollar? There are 100 cents in a dollar. How many years are in a century? There are 100 years in a century. Does this give you a clue about what the word “percent” means? It is really two words, “per cent,” and means per one hundred. A **percent** is a ratio whose denominator is 100. We use the percent symbol $\%$, to show percent.

Note:
Percent
A percent is a ratio whose denominator is 100.

According to data from the American Association of Community Colleges (2015), about 57% of community college students are female. This means 57 out of every 100 community college students are female, as [link] shows. Out of the 100 squares on the grid, 57 are shaded, which we write as the ratio $\frac{57}{100}$.



Among every 100 community college students, 57 are female.

Similarly, 25% means a ratio of $\frac{25}{100}$, 3% means a ratio of $\frac{3}{100}$ and 100% means a ratio of $\frac{100}{100}$. In words, "one hundred percent" means the total 100% is $\frac{100}{100}$, and since $\frac{100}{100} = 1$, we see that 100% means 1 whole.

Example:
Exercise:

Problem:

According to the Public Policy Institute of California (2010), 44% of parents of public school children would like their youngest child to earn a graduate degree. Write this percent as a ratio.

Solution:
Solution

The amount we want to convert is 44%.	44%
Write the percent as a ratio. Remember that *percent* means per 100.	$\frac{44}{100}$

Note:
Exercise:

Problem: Write the percent as a ratio.

According to a survey, 89% of college students have a smartphone.

Solution:

$\frac{89}{100}$

Note:
Exercise:

Problem: Write the percent as a ratio.

A study found that 72% of U.S. teens send text messages regularly.

Solution:

$\frac{72}{100}$

Example:
Exercise:

Problem:

In 2007, according to a U.S. Department of Education report, 21 out of every 100 first-time freshmen college students at 4-year public institutions took at least one remedial course. Write this as a ratio and then as a percent.

Solution:
Solution

The amount we want to convert is 21 out of 100.	21 out of 100
Write as a ratio.	$\frac{21}{100}$
Convert the 21 per 100 to percent.	21%

Note:
Exercise:

Problem:

Write as a ratio and then as a percent: The American Association of Community Colleges reported that 62 out of 100 full-time community college students balance their studies with full-time or part time employment.

Solution:

$\frac{62}{100}, 62\%$

Note:
Exercise:

Problem:

Write as a ratio and then as a percent: In response to a student survey, 41 out of 100 Santa Ana College students expressed a goal of earning an Associate's degree or transferring to a four-year college.

Solution:

$\frac{41}{100}, 41\%$

Convert Percents to Fractions and Decimals

Since percents are ratios, they can easily be expressed as fractions. Remember that percent means per 100, so the denominator of the fraction is 100.

Note:

Convert a percent to a fraction.

Write the percent as a ratio with the denominator 100.
Simplify the fraction if possible.

Example:
Exercise:

Problem: Convert each percent to a fraction:

ⓐ 36%
ⓑ 125%

Solution:
Solution

ⓐ	
	36%
Write as a ratio with denominator 100.	$\frac{36}{100}$
Simplify.	$\frac{9}{25}$

ⓑ	
	125%
Write as a ratio with denominator 100.	$\frac{125}{100}$
Simplify.	$\frac{5}{4}$

Note:
Exercise:

Problem: Convert each percent to a fraction:

ⓐ 48%
ⓑ 110%

Solution:

ⓐ $\frac{12}{25}$
ⓑ $\frac{11}{10}$

Note:
Exercise:

Problem: Convert each percent to a fraction:

ⓐ 64%
ⓑ 150%

Solution:

ⓐ $\frac{16}{25}$
ⓑ $\frac{3}{2}$

The previous example shows that a percent can be greater than 1. We saw that 125% means $\frac{125}{100}$, or $\frac{5}{4}$. These are improper fractions, and their values are greater than one.

Example:
Exercise:

Problem: Convert each percent to a fraction:

ⓐ 24.5%
ⓑ $33\frac{1}{3}\%$

Solution:
Solution

ⓐ	
	24.5%

Write as a ratio with denominator 100.	$\frac{24.5}{100}$
Clear the decimal by multiplying numerator and denominator by 10.	$\frac{24.5(10)}{100(10)}$
Multiply.	$\frac{245}{1000}$
Rewrite showing common factors.	$\frac{5\cdot 49}{5\cdot 200}$
Simplify.	$\frac{49}{200}$

ⓑ	
	$33\frac{1}{3}\%$
Write as a ratio with denominator 100.	$\frac{33\frac{1}{3}}{100}$
Write the numerator as an improper fraction.	$\frac{\frac{100}{3}}{100}$
Rewrite as fraction division, replacing 100 with $\frac{100}{1}$.	$\frac{100}{3} \div \frac{100}{1}$
Multiply by the reciprocal.	$\frac{100}{3} \cdot \frac{1}{100}$
Simplify.	$\frac{1}{3}$

Note:
Exercise:

Problem: Convert each percent to a fraction:

ⓐ 64.4%
ⓑ $66\frac{2}{3}\%$

Solution:

ⓐ $\frac{161}{250}$
ⓑ $\frac{2}{3}$

Note:
Exercise:

Problem: Convert each percent to a fraction:

ⓐ 42.5%
ⓑ $8\frac{3}{4}\%$

Solution:

ⓐ $\frac{113}{250}$
ⓑ $\frac{7}{80}$

In Decimals, we learned how to convert fractions to decimals. To convert a percent to a decimal, we first convert it to a fraction and then change the fraction to a decimal.

Note:
Convert a percent to a decimal.

Write the percent as a ratio with the denominator 100.
Convert the fraction to a decimal by dividing the numerator by the denominator.

Example:
Exercise:

Problem: Convert each percent to a decimal:

ⓐ 6%
ⓑ 78%

Solution:
Solution

Because we want to change to a decimal, we will leave the fractions with denominator 100 instead of removing common factors.

ⓐ	
	6%

Write as a ratio with denominator 100.	$\frac{6}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.06

ⓑ	
	78%
Write as a ratio with denominator 100.	$\frac{78}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.78

Note:
Exercise:

Problem: Convert each percent to a decimal:

ⓐ 9%
ⓑ 87%

Solution:

ⓐ 0.09
ⓑ 0.87

Note:
Exercise:

Problem: Convert each percent to a decimal:

ⓐ 3%
ⓑ 91%

Solution:

ⓐ 0.03
ⓑ 0.91

Example:
Exercise:

Problem: Convert each percent to a decimal:

ⓐ 135%
ⓑ 12.5%

Solution:
Solution

ⓐ	
	135%
Write as a ratio with denominator 100.	$\frac{135}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	1.35

ⓑ	
	12.5%
Write as a ratio with denominator 100.	$\frac{12.5}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.125

Note:
Exercise:

Problem: Convert each percent to a decimal:

ⓐ 115%
ⓑ 23.5%

Solution:

ⓐ 1.15
ⓑ 0.235

Note:
Exercise:

Problem: Convert each percent to a decimal:

ⓐ 123%
ⓑ 16.8%

Solution:

ⓐ 1.23
ⓑ 0.168

Let's summarize the results from the previous examples in [link], and look for a pattern we could use to quickly convert a percent number to a decimal number.

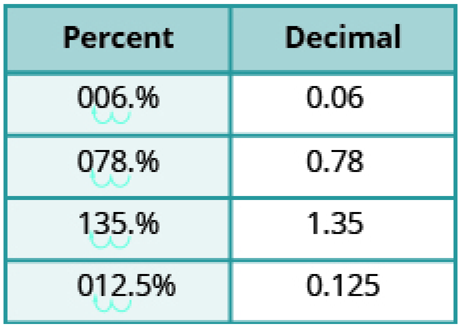
Percent	**Decimal**
6%	0.06
78%	0.78

Percent	**Decimal**
135%	1.35
12.5%	0.125

Do you see the pattern?

To convert a percent number to a decimal number, we move the decimal point two places to the left and remove the $\%$ sign. (Sometimes the decimal point does not appear in the percent number, but just like we can think of the integer 6 as 6.0, we can think of 6% as 6.0%.) Notice that we may need to add zeros in front of the number when moving the decimal to the left.

[link] uses the percents in [link] and shows visually how to convert them to decimals by moving the decimal point two places to the left.



Percent	Decimal
006.%	0.06
078.%	0.78
135.%	1.35
012.5%	0.125

Example:
Exercise:

Problem:

Among a group of business leaders, 77% believe that poor math and science education in the U.S. will lead to higher unemployment rates.

Convert the percent to: ⓐ a fraction ⓑ a decimal

Solution:
Solution

ⓐ	
	77%
Write as a ratio with denominator 100.	$\frac{77}{100}$

ⓑ	
	$\frac{77}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.77

Note:
Exercise:

Problem: Convert the percent to: ⓐ a fraction and ⓑ a decimal

Twitter's share of web traffic jumped 24% when one celebrity tweeted live on air.

Solution:

ⓐ $\frac{6}{25}$
ⓑ 0.24

Note:
Exercise:

Problem: Convert the percent to: ⓐ a fraction and ⓑ a decimal

The U.S. Census estimated that in 2013, 44% of the population of Boston age 25 or older have a bachelor's or higher degrees.

Solution:

ⓐ $\frac{22}{50}$
ⓑ 0.44

Example:
Exercise:

Problem:

There are four suits of cards in a deck of cards—hearts, diamonds, clubs, and spades. The probability of randomly choosing a heart from a shuffled deck of cards is 25%. Convert the percent to:

ⓐ a fraction

ⓑ a decimal

(credit: Riles32807, Wikimedia Commons)

Solution:
Solution

ⓐ	
	25%
Write as a ratio with denominator 100.	$\frac{25}{100}$
Simplify.	$\frac{1}{4}$

ⓑ	$\frac{1}{4}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.25

Note:
Exercise:

Problem: Convert the percent to: ⓐ a fraction, and ⓑ a decimal

The probability that it will rain Monday is 30%.

Solution:

ⓐ $\frac{3}{10}$
ⓑ 0.3

Note:
Exercise:

Problem: Convert the percent to: ⓐ a fraction, and ⓑ a decimal

The probability of getting heads three times when tossing a coin three times is 12.5%.

Solution:

ⓐ $\frac{12.5}{100}$
ⓑ 0.125

Convert Decimals and Fractions to Percents

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent.

Note:
Convert a decimal to a percent.

Write the decimal as a fraction.
If the denominator of the fraction is not 100, rewrite it as an equivalent fraction with denominator 100.
Write this ratio as a percent.

Example:
Exercise:

Problem: Convert each decimal to a percent: ⓐ 0.05 ⓑ 0.83

Solution:
Solution

ⓐ	

	0.05
Write as a fraction. The denominator is 100.	$\frac{5}{100}$
Write this ratio as a percent.	5%

ⓑ	
	0.83
The denominator is 100.	$\frac{83}{100}$
Write this ratio as a percent.	83%

Note:
Exercise:

Problem: Convert each decimal to a percent: ⓐ 0.01 ⓑ 0.17.

Solution:

ⓐ 1%
ⓑ 17%

Note:
Exercise:

Problem: Convert each decimal to a percent: ⓐ 0.04 ⓑ 0.41

Solution:

ⓐ 4%
ⓑ 41%

To convert a mixed number to a percent, we first write it as an improper fraction.

Example:
Exercise:

Problem: Convert each decimal to a percent: ⓐ 1.05 ⓑ 0.075

Solution:
Solution

ⓐ	
	0.05
Write as a fraction.	$1\frac{5}{100}$

Write as an improper fraction. The denominator is 100.	$\frac{105}{100}$
Write this ratio as a percent.	105%

Notice that since $1.05 > 1$, the result is more than 100%.

ⓑ	
	0.075
Write as a fraction. The denominator is 1,000.	$\frac{75}{1{,}000}$
Divide the numerator and denominator by 10, so that the denominator is 100.	$\frac{7.5}{100}$
Write this ratio as a percent.	7.5%

Note:
Exercise:

Problem: Convert each decimal to a percent: ⓐ 1.75 ⓑ 0.0825

Solution:

ⓐ 175%
ⓑ 8.25%

Note:
Exercise:

Problem: Convert each decimal to a percent: ⓐ 2.25 ⓑ 0.0925

Solution:

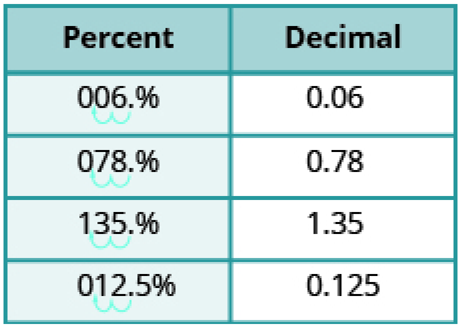
ⓐ 225%
ⓑ 9.25%

Let's summarize the results from the previous examples in [link] so we can look for a pattern.

Decimal	Percent
0.05	5%
0.83	83%
1.05	105%
0.075	7.5%

Do you see the pattern? To convert a decimal to a percent, we move the decimal point two places to the right and then add the percent sign.

[link] uses the decimal numbers in [link] and shows visually to convert them to percents by moving the decimal point two places to the right and then writing the % sign.



Percent	Decimal
006.%	0.06
078.%	0.78
135.%	1.35
012.5%	0.125

In Decimals, we learned how to convert fractions to decimals. Now we also know how to change decimals to percents. So to convert a fraction to a percent, we first change it to a decimal and then convert that decimal to a percent.

Note:
Convert a fraction to a percent.

Convert the fraction to a decimal.
Convert the decimal to a percent.

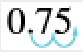
Example:
Exercise:

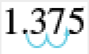
Problem:

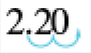
Convert each fraction or mixed number to a percent: ⓐ $\frac{3}{4}$ ⓑ $\frac{11}{8}$ ⓒ $2\frac{1}{5}$

Solution:
Solution

To convert a fraction to a decimal, divide the numerator by the denominator.

ⓐ	
Change to a decimal.	$\frac{3}{4}$
Write as a percent by moving the decimal two places.	
	75%

ⓑ	
Change to a decimal.	$\frac{11}{8}$
Write as a percent by moving the decimal two places.	
	137.5%

ⓒ	
Write as an improper fraction.	$2\frac{1}{5}$
Change to a decimal.	$\frac{11}{5}$
Write as a percent.	 2.20
	220%

Notice that we needed to add zeros at the end of the number when moving the decimal two places to the right.

Note:
Exercise:

Problem:

Convert each fraction or mixed number to a percent: ⓐ $\frac{5}{8}$ ⓑ $\frac{11}{4}$ ⓒ $3\frac{2}{5}$

Solution:

ⓐ 62.5%
ⓑ 275%
ⓒ 340%

Note:
Exercise:

Problem:

Convert each fraction or mixed number to a percent: ⓐ $\frac{7}{8}$ ⓑ $\frac{9}{4}$ ⓒ $1\frac{3}{5}$

Solution:

ⓐ 87.5%
ⓑ 225%
ⓒ 160%

Sometimes when changing a fraction to a decimal, the division continues for many decimal places and we will round off the quotient. The number of decimal places we round to will depend on the situation. If the decimal involves money, we round to the hundredths place. For most other cases in this book we will round the number to the nearest thousandth, so the percent will be rounded to the nearest tenth.

Example:
Exercise:

Problem: Convert $\frac{5}{7}$ to a percent.

Solution:
Solution

To change a fraction to a decimal, we divide the numerator by the denominator.

	$\frac{5}{7}$
Change to a decimal—rounding to the nearest thousandth.	0.714
Write as a percent.	71.4%

Note:
Exercise:

Problem: Convert the fraction to a percent: $\frac{3}{7}$

Solution:

42.9%

Note:
Exercise:

Problem: Convert the fraction to a percent: $\frac{4}{7}$

Solution:

57.1%

When we first looked at fractions and decimals, we saw that fractions converted to a repeating decimal. When we converted the fraction $\frac{4}{3}$ to a decimal, we wrote the answer as 1.3. We will use this same notation, as well as fraction notation, when we convert fractions to percents in the next example.

Example:
Exercise:

Problem:

An article in a medical journal claimed that approximately $\frac{1}{3}$ of American adults are obese. Convert the fraction $\frac{1}{3}$ to a percent.

Solution:
Solution

	$\frac{1}{3}$
Change to a decimal.	$\begin{array}{r} 0.33\ldots \\ 3\overline{)1.00} \\ \underline{9} \\ 10 \\ \underline{9} \\ 1 \end{array}$
Write as a repeating decimal.	$0.333\ldots$
Write as a percent.	$33\frac{1}{3}\%$

We could also write the percent as $33.\overline{3}\%$.

Note:
Exercise:

Problem: Convert the fraction to a percent:

According to the U.S. Census Bureau, about $\frac{1}{9}$ of United States housing units have just 1 bedroom.

Solution:

$11.\overline{1}\%$, or $11\frac{1}{9}\%$

Note:
Exercise:

Problem: Convert the fraction to a percent:

According to the U.S. Census Bureau, about $\frac{1}{6}$ of Colorado residents speak a language other than English at home.

Solution:

$16.\overline{6}\%$, or $16\frac{2}{3}\%$

Key Concepts

- **Convert a percent to a fraction.**

Write the percent as a ratio with the denominator 100.
Simplify the fraction if possible.

- **Convert a percent to a decimal.**

  Write the percent as a ratio with the denominator 100.
  Convert the fraction to a decimal by dividing the numerator by the denominator.

- **Convert a decimal to a percent.**

  Write the decimal as a fraction.
  If the denominator of the fraction is not 100, rewrite it as an equivalent fraction with denominator 100.
  Write this ratio as a percent.

- **Convert a fraction to a percent.**

  Convert the fraction to a decimal.
  Convert the decimal to a percent.

Practice Makes Perfect

Use the Definition of Percents

In the following exercises, write each percent as a ratio.

Exercise:

Problem:

In 2014, the unemployment rate for those with only a high school degree was 6.0%.

Solution:

$\frac{6}{100}$

Exercise:

Problem:

In 2015, among the unemployed, 29% were long-term unemployed.

Exercise:

Problem:

The unemployment rate for those with Bachelor's degrees was 3.2% in 2014.

Solution:

$\frac{32}{1000}$

Exercise:

Problem: The unemployment rate in Michigan in 2014 was 7.3%.

In the following exercises, write as

ⓐ a ratio and
ⓑ a percent

Exercise:

Problem:

57 out of 100 nursing candidates received their degree at a community college.

Solution:

ⓐ $\frac{57}{100}$
ⓑ 57%

Exercise:

Problem:

80 out of 100 firefighters and law enforcement officers were educated at a community college.

Exercise:

Problem:

42 out of 100 first-time freshmen students attend a community college.

Solution:

ⓐ $\frac{42}{100}$
ⓑ 42%

Exercise:

Problem:

71 out of 100 full-time community college faculty have a master's degree.

Convert Percents to Fractions and Decimals

In the following exercises, convert each percent to a fraction and simplify all fractions.

Exercise:

Problem: 4%

Solution:

$\frac{1}{25}$

Exercise:

Problem: 8%

Exercise:

Problem: 17%

Solution:

$\frac{17}{100}$

Exercise:

Problem: 19%

Exercise:

Problem: 52%

Solution:

$\frac{13}{25}$

Exercise:

Problem: 78%

Exercise:

Problem: 125%

Solution:

$\frac{5}{4}$

Exercise:

Problem: 135%

Exercise:

Problem: 37.5%

Solution:

$\frac{3}{8}$

Exercise:

Problem: 42.5%

Exercise:

Problem: 18.4%

Solution:

$\frac{23}{125}$

Exercise:

Problem: 46.4%

Exercise:

Problem: $9\frac{1}{2}\%$

Solution:

$\frac{19}{200}$

Exercise:

Problem: $8\frac{1}{2}\%$

Exercise:

Problem: $5\frac{1}{3}\%$

Solution:

$\frac{4}{75}$

Exercise:

Problem: $6\frac{2}{3}\%$

In the following exercises, convert each percent to a decimal.

Exercise:

Problem: 5%

Solution:

0.05

Exercise:

Problem: 9%

Exercise:

Problem: 1%

Solution:

0.01

Exercise:

Problem: 2%

Exercise:

Problem: 63%

Solution:

0.63

Exercise:

Problem: 71%

Exercise:

Problem: 40%

Solution:

0.4

Exercise:

Problem: 50%

Exercise:

Problem: 115%

Solution:

1.15

Exercise:

Problem: 125%

Exercise:

Problem: 150%

Solution:

1.5

Exercise:

Problem: 250%

Exercise:

Problem: 21.4%

Solution:

0.214

Exercise:

Problem: 39.3%

Exercise:

Problem: 7.8%

Solution:

0.078

Exercise:

Problem: 6.4%

In the following exercises, convert each percent to

ⓐ a simplified fraction and
ⓑ a decimal

Exercise:

Problem:

In 2010, 1.5% of home sales had owner financing. (*Source:* Bloomberg Businessweek, 5/23–29/2011)

Solution:

ⓐ $\frac{3}{200}$
ⓑ 0.015

Exercise:

Problem:

In 2000, 4.2% of the United States population was of Asian descent. (*Source:* www.census.gov)

Exercise:

Problem:

According to government data, in 2013 the number of cell phones in India was 70.23% of the population.

Solution:

ⓐ $\frac{7023}{10{,}000}$
ⓑ 0.7023

Exercise:

Problem:

According to the U.S. Census Bureau, among Americans age 25 or older who had doctorate degrees in 2014, 37.1% are women.

Exercise:

Problem:

A couple plans to have two children. The probability they will have two girls is 25%.

Solution:

ⓐ $\frac{1}{4}$
ⓑ 0.25

Exercise:

Problem:

Javier will choose one digit at random from 0 through 9. The probability he will choose 3 is 10%.

Exercise:

Problem:

According to the local weather report, the probability of thunderstorms in New York City on July 15 is 60%.

Solution:

ⓐ $\frac{3}{5}$
ⓑ 0.6

Exercise:

Problem:

A club sells 50 tickets to a raffle. Osbaldo bought one ticket. The probability he will win the raffle is 2%.

Convert Decimals and Fractions to Percents

In the following exercises, convert each decimal to a percent.

Exercise:

Problem: 0.01

Solution:

1%

Exercise:

Problem: 0.03

Exercise:

Problem: 0.18

Solution:

18%

Exercise:

Problem: 0.15

Exercise:

Problem: 1.35

Solution:

135%

Exercise:

Problem: 1.56

Exercise:

Problem: 3

Solution:

300%

Exercise:

Problem: 4

Exercise:

Problem: 0.009

Solution:

0.9%

Exercise:

Problem: 0.008

Exercise:

Problem: 0.0875

Solution:

8.75%

Exercise:

Problem: 0.0625

Exercise:

Problem: 1.5

Solution:

150%

Exercise:

Problem: 2.2

Exercise:

Problem: 2.254

Solution:

225.4%

Exercise:

Problem: 2.317

In the following exercises, convert each fraction to a percent.

Exercise:

Problem: $\frac{1}{4}$

Solution:

25%

Exercise:

Problem: $\frac{1}{5}$

Exercise:

Problem: $\frac{3}{8}$

Solution:

37.5%

Exercise:

Problem: $\frac{5}{8}$

Exercise:

Problem: $\frac{7}{4}$

Solution:

175%

Exercise:

Problem: $\frac{9}{8}$

Exercise:

Problem: $6\frac{4}{5}$

Solution:

680%

Exercise:

Problem: $5\frac{1}{4}$

Exercise:

Problem: $\frac{5}{12}$

Solution:

41.7%

Exercise:

Problem: $\frac{11}{12}$

Exercise:

Problem: $2\frac{2}{3}$

Solution:

$266.\bar{6}\%$

Exercise:

Problem: $1\frac{2}{3}$

Exercise:

Problem: $\frac{3}{7}$

Solution:

42.9%

Exercise:

Problem: $\frac{6}{7}$

Exercise:

Problem: $\frac{5}{9}$

Solution:

55.6%

Exercise:

Problem: $\frac{4}{9}$

In the following exercises, convert each fraction to a percent.

Exercise:

Problem: $\frac{1}{4}$ of washing machines needed repair.

Solution:

25%

Exercise:

Problem: $\frac{1}{5}$ of dishwashers needed repair.

In the following exercises, convert each fraction to a percent.

Exercise:

Problem:

According to the National Center for Health Statistics, in 2012, $\frac{7}{20}$ of American adults were obese.

Solution:

35%

Exercise:

Problem:

The U.S. Census Bureau estimated that in 2013, 85% of Americans lived in the same house as they did 1 year before.

In the following exercises, complete the table.

Exercise:

Problem:

Fraction	Decimal	Percent
$\frac{1}{2}$		
	0.45	
		18%
$\frac{1}{3}$		
	0.0008	
2		

Exercise:

Problem:

Fraction	Decimal	Percent
$\frac{1}{4}$		
	0.65	

Fraction	**Decimal**	**Percent**
		22%
$\frac{2}{3}$		
	0.0004	
3		

Everyday Math

Exercise:

Problem:

Sales tax Felipa says she has an easy way to estimate the sales tax when she makes a purchase. The sales tax in her city is 9.05%. She knows this is a little less than 10%.

ⓐ Convert 10% to a fraction.
ⓑ Use your answer from ⓐ to estimate the sales tax Felipa would pay on a $95 dress.

Solution:

ⓐ $\frac{1}{10}$
ⓑ approximately $\$9.50$

Exercise:

Problem:

Savings Ryan has 25% of each paycheck automatically deposited in his savings account.

ⓐ Write 25% as a fraction.
ⓑ Use your answer from ⓐ to find the amount that goes to savings from Ryan's $\$2,400$ paycheck.

Exercise:

Problem:

Amelio is shopping for textbooks online. He found three sellers that are offering a book he needs for the same price, including shipping. To decide which seller to buy from he is comparing their customer satisfaction ratings. The ratings are given in the chart.

Seller	**Rating**
A	$4/5$
B	$3.5/4$
C	85%

Exercise:

Problem: Write seller C’s rating as a fraction and a decimal.

Solution:

$\frac{17}{20};\ 0.85$

Exercise:

Problem: Write seller B’s rating as a percent and a decimal.

Exercise:

Problem: Write seller A’s rating as a percent and a decimal.

Solution:

80%; 0.8

Exercise:

Problem: Which seller should Amelio buy from and why?

Writing Exercises

Exercise:

Problem:

Convert $25\%, 50\%, 75\%,$ and 100% to fractions. Do you notice a pattern? Explain what the pattern is.

Solution:

$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1.$

Exercise:

Problem:

Convert $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}$, and $\frac{9}{10}$ to percents. Do you notice a pattern? Explain what the pattern is.

Exercise:

Problem:

When the Szetos sold their home, the selling price was 500% of what they had paid for the house 30 years ago. Explain what 500% means in this context.

Solution:

The Szetos sold their home for five times what they paid 30 years ago.

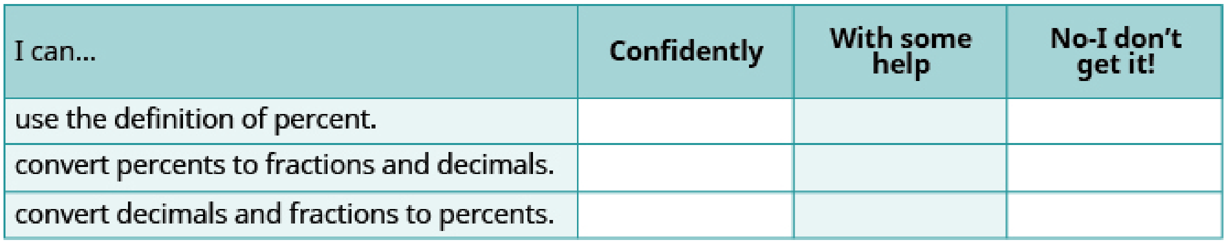
Exercise:

Problem:

According to cnn.com, cell phone use in 2008 was 600% of what it had been in 2001. Explain what 600% means in this context.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
use the definition of percent.			
convert percents to fractions and decimals.			
convert decimals and fractions to percents.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don’t get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

percent
A perfecnt is a ratio whose denominator is 100.

Solve General Applications of Percent

By the end of this section, you will be able to:

- Translate and solve basic percent equations
- Solve applications of percent
- Find percent increase and percent decrease

Note:
Before you get started, take this readiness quiz.

1. Translate and solve: $\frac{3}{4}$ of x is 24.
   If you missed this problem, review [link].
2. Simplify: $(4.5)\,(2.38)$.
   If you missed this problem, review [link].
3. Solve: $3.5 = 0.7n$.
   If you missed this problem, review [link].

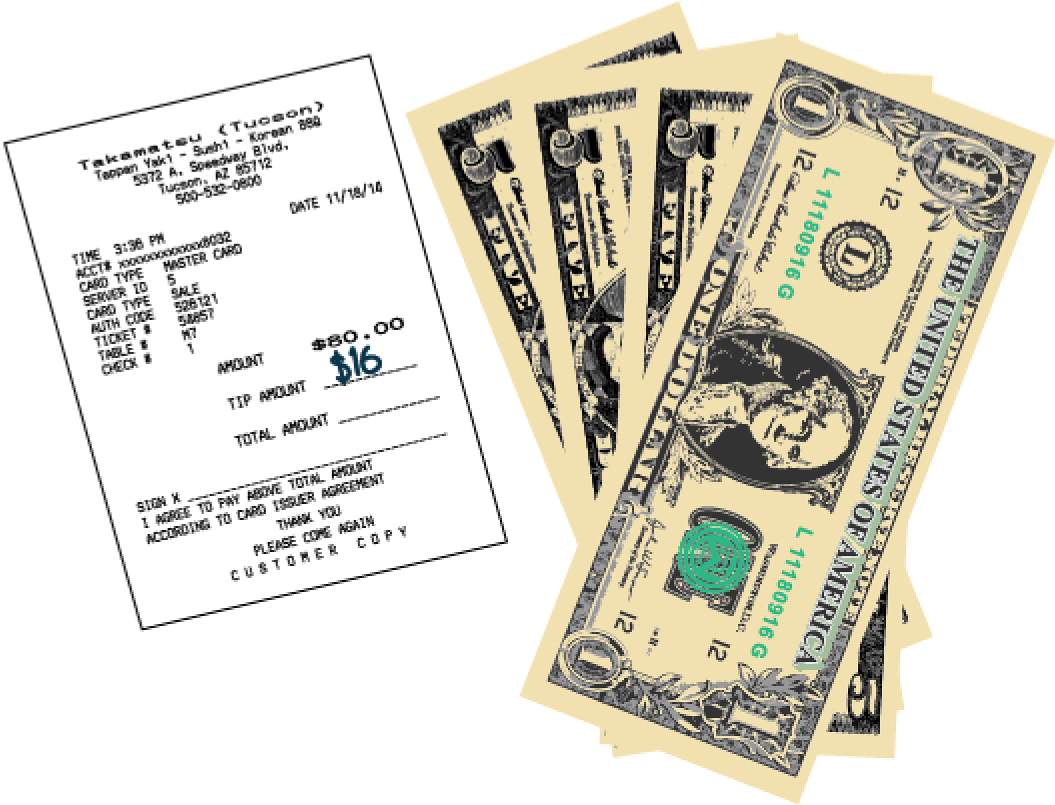
Translate and Solve Basic Percent Equations

We will solve percent equations by using the methods we used to solve equations with fractions or decimals. In the past, you may have solved percent problems by setting them up as proportions. That was the best method available when you did not have the tools of algebra. Now as a prealgebra student, you can translate word sentences into algebraic equations, and then solve the equations.

We'll look at a common application of percent—tips to a server at a restaurant—to see how to set up a basic percent application.

When Aolani and her friends ate dinner at a restaurant, the bill came to $\$80$. They wanted to leave a 20% tip. What amount would the tip be?

To solve this, we want to find what *amount* is 20% of $\$80$. The $\$80$ is called the *base*. The amount of the tip would be $0.20\left(80\right)$, or $\$16$ See [link]. To find the amount of the tip, we multiplied the percent by the base.


Takamatsu (Tucson)
Tappan Yaki - Sushi - Korean BBQ
5372 A. Speedway Blvd.
Tucson, AZ 85712
500-532-0800
DATE 11/18/14
TIME 3:36 PM
ACCT# xxxxxxxxxxxx8032
CARD TYPE MASTER CARD
SERVER ID 5
CARD TYPE SALE
AUTH CODE 526121
TICKET # 58857
TABLE # M7
CHECK # 1
AMOUNT $80.00
TIP AMOUNT $16
TOTAL AMOUNT
SIGN X
I AGREE TO PAY ABOVE TOTAL AMOUNT
ACCORDING TO CARD ISSUER AGREEMENT
THANK YOU
PLEASE COME AGAIN
CUSTOMER COPY
THE UNITED STATES OF AMERICA
ONE DOLLAR
FIVE
L 11180916 G
12

A 20% tip for an $\$80$ restaurant bill comes out to $\$16$.

In the next examples, we will find the amount. We must be sure to change the given percent to a decimal when we translate the words into an equation.

Example:
Exercise:

Problem: What number is 35% of 90?

Solution:
Solution

Translate into algebra. Let $n =$ the number.	$\underbrace{\text{What number}}_{n}\ \underbrace{\text{is}}_{=}\ \underbrace{35\%}_{0.35}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{90?}_{90}$
Multiply.	$n = 31.5$
	31.5 is 35% of 90

Note:
Exercise:

Problem: What number is 45% of 80?

Solution:

36

Note:
Exercise:

Problem: What number is 55% of 60?

Solution:

33

Example:
Exercise:

Problem: 125% of 28 is what number?

Solution:
Solution

Translate into algebra. Let $a=$ the number.	$\underbrace{125\%}_{1.25}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{28}_{28}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{what number?}}_{a}$
Multiply.	$35 = a$
	125% of 28 is 35.

Remember that a percent over 100 is a number greater than 1. We found that 125% of 28 is 35, which is greater than 28.

Note:
Exercise:

Problem: 150% of 78 is what number?

Solution:

117

Note:
Exercise:

Problem: 175% of 72 is what number?

Solution:

126

In the next examples, we are asked to find the base.

Example:
Exercise:

Problem: Translate and solve: 36 is 75% of what number?

Solution:
Solution

Translate. Let $b =$ the number.	$\underbrace{36}_{36}\ \underbrace{\text{is}}_{=}\ \underbrace{75\%}_{0.75}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{\text{what number?}}_{b}$
Divide both sides by 0.75.	$\frac{36}{0.75} = \frac{0.75b}{0.75}$
Simplify.	$48 = b$ 36 is 75% of 48.

Note:
Exercise:

Problem: 17 is 25% of what number?

Solution:

68

Note:
Exercise:

Problem: 40 is 62.5% of what number?

Solution:

64

Example:
Exercise:

Problem: 6.5% of what number is $\$1.17$?

Solution:
Solution

Translate. Let $b =$ the number.	$\underbrace{6.5\%}_{0.065}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{\text{what number}}_{b}\ \underbrace{\text{is}}_{=}\ \underbrace{\$1.17?}_{1.17}$
Divide both sides by 0.065.	$\frac{0.065n}{0.065} = \frac{1.17}{0.065}$
Simplify.	$n = 18$ 6.5% of \$18 is \$1.17.

Note:
Exercise:

Problem: 7.5% of what number is \$1.95?

Solution:

\$26

Note:
Exercise:

Problem: 8.5% of what number is \$3.06?

Solution:

\$36

In the next examples, we will solve for the percent.

Example:
Exercise:

Problem: What percent of 36 is 9?

Solution:
Solution

Translate into algebra. Let $p =$ the percent.	$\underbrace{\text{What percent}}_{p}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{36}_{36}\ \underbrace{\text{is}}_{=}\ \underbrace{9?}_{9}$
Divide by 36.	$\frac{36p}{36} = \frac{9}{36}$
Simplify.	$p = \frac{1}{4}$
Convert to decimal form.	$p = 0.25$
Convert to percent.	$p = 25\%$ 25% of 36 is 9.

Note:
Exercise:

Problem: What percent of 76 is 57?

Solution:

75%

Note:
Exercise:

Problem: What percent of 120 is 96?

Solution:

80%

Example:
Exercise:

Problem: 144 is what percent of 96?

Solution:
Solution

Translate into algebra. Let $p =$ the percent.	$\underbrace{144}_{144}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{what percent}}_{p}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{96?}_{96}$
Divide by 96.	

	$\frac{144}{96} = \frac{96p}{96}$
Simplify.	$1.5 = p$
Convert to percent.	$150\% = p$ 144 is 150% of 96.

Note:
Exercise:

Problem: 110 is what percent of 88?

Solution:

125%

Note:
Exercise:

Problem: 126 is what percent of 72?

Solution:

175%

Solve Applications of Percent

Many applications of percent occur in our daily lives, such as tips, sales tax, discount, and interest. To solve these applications we'll translate to a basic percent equation, just like those we solved in the previous examples in this section. Once you translate the sentence into a percent equation, you know how to solve it.

We will update the strategy we used in our earlier applications to include equations now. Notice that we will translate a sentence into an equation.

Note:
Solve an application

Identify what you are asked to find and choose a variable to represent it.
Write a sentence that gives the information to find it.
Translate the sentence into an equation.
Solve the equation using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Write a complete sentence that answers the question.

Now that we have the strategy to refer to, and have practiced solving basic percent equations, we are ready to solve percent applications. Be sure to ask yourself if your final answer makes sense—since many of the applications we'll solve involve everyday situations, you can rely on your own experience.

Example:
Exercise:

Problem:

Dezohn and his girlfriend enjoyed a dinner at a restaurant, and the bill was $\$68.50$. They want to leave an 18% tip. If the tip will be 18% of the total bill, how much should the tip be?

Solution:
Solution

What are you asked to find?	the amount of the tip
Choose a variable to represent it.	Let $t =$ amount of tip.
Write a sentence that give the information to find it.	The tip is 18% of the total bill.
Translate the sentence into an equation.	$\underbrace{\text{The tip}}_{t}\ \underbrace{\text{is}}_{=}\ \underbrace{18\%}_{0.18}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{\$68.50.}_{\$68.50}$
Multiply.	$t = 12.33$
Check. Is this answer reasonable?	
If we approximate the bill to $70 and the percent to 20%, we would have a tip of $14. So a tip of $12.33 seems reasonable.	

Write a complete sentence that answers the question.	The couple should leave a tip of $12.33.

Note:
Exercise:

Problem:

Cierra and her sister enjoyed a special dinner in a restaurant, and the bill was $81.50. If she wants to leave 18% of the total bill as her tip, how much should she leave?

Solution:

$14.67

Note:
Exercise:

Problem:

Kimngoc had lunch at her favorite restaurant. She wants to leave 15% of the total bill as her tip. If her bill was $14.40, how much will she leave for the tip?

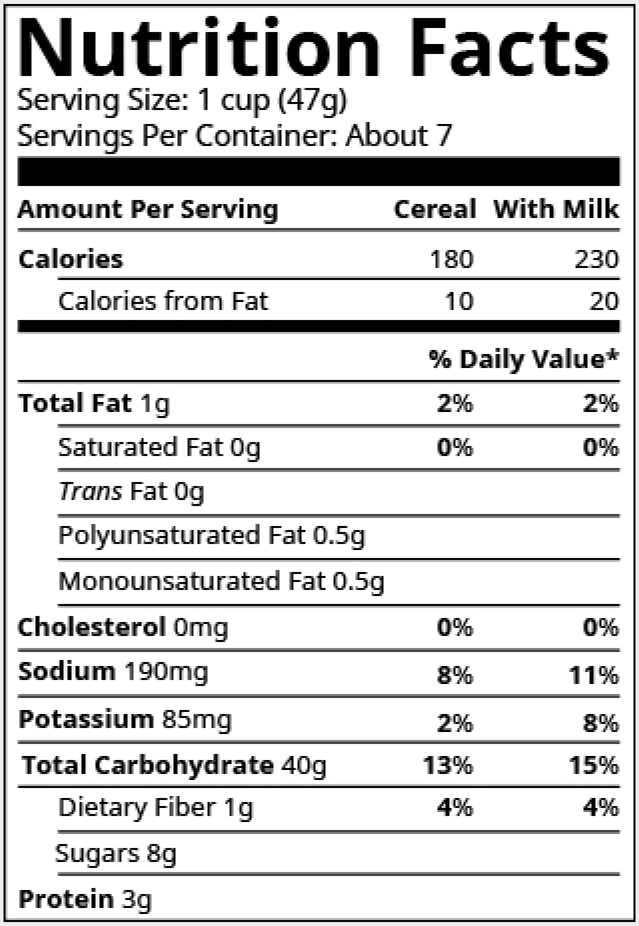
Solution:

$2.16

Example:
Exercise:

Problem:

The label on Masao's breakfast cereal said that one serving of cereal provides 85 milligrams (mg) of potassium, which is 2% of the recommended daily amount. What is the total recommended daily amount of potassium?



Nutrition Facts

Serving Size: 1 cup (47g)
Servings Per Container: About 7

Amount Per Serving	**Cereal**	**With Milk**
Calories	180	230
Calories from Fat	10	20
	% Daily Value*	
Total Fat 1g	**2%**	**2%**
Saturated Fat 0g	**0%**	**0%**
Trans Fat 0g		
Polyunsaturated Fat 0.5g		
Monounsaturated Fat 0.5g		
Cholesterol 0mg	**0%**	**0%**
Sodium 190mg	**8%**	**11%**
Potassium 85mg	**2%**	**8%**
Total Carbohydrate 40g	**13%**	**15%**
Dietary Fiber 1g	**4%**	**4%**
Sugars 8g		
Protein 3g		

Solution:
Solution

What are you asked to find?	the total amount of potassium recommended
Choose a variable to represent it.	Let a = total amount of potassium.
Write a sentence that gives the information to find it.	85 mg is 2% of the total amount.
Translate the sentence into an equation.	$\underbrace{\text{85 mg}}_{85}\ \underbrace{\text{is}}_{=}\ \underbrace{2\%}_{0.02}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{a}_{a}\,?$
Divide both sides by 0.02.	$\frac{85}{0.02}=\frac{0.02a}{0.02}$
Simplify.	$4{,}250 = a$
Check: Is this answer reasonable?	
Yes. 2% is a small percent and 85 is a small part of 4,250.	
Write a complete sentence that answers the question.	The amount of potassium that is recommended is 4250 mg.

Note:
Exercise:

Problem:

One serving of wheat square cereal has 7 grams of fiber, which is 29% of the recommended daily amount. What is the total recommended daily amount of fiber?

Solution:

24.1 grams

Note:
Exercise:

Problem:

One serving of rice cereal has 190 mg of sodium, which is 8% of the recommended daily amount. What is the total recommended daily amount of sodium?

Solution:

2,375 mg

Example:
Exercise:

Problem:

Mitzi received some gourmet brownies as a gift. The wrapper said each brownie was 480 calories, and had 240 calories of fat. What percent of the total calories in each brownie comes from fat?

Solution:
Solution

What are you asked to find?	the percent of the total calories from fat
Choose a variable to represent it.	Let $p =$ percent from fat.
Write a sentence that gives the information to find it.	What percent of 480 is 240?
Translate the sentence into an equation.	$\underbrace{\text{What percent}}_{p}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{480}_{480}\ \underbrace{\text{is}}_{=}\ \underbrace{240?}_{240}$
Divide both sides by 480.	$\frac{p \cdot 480}{480} = \frac{240}{480}$
Simplify.	$p = 0.5$
Convert to percent form.	$p = 50\%$
Check. Is this answer reasonable?	

Yes. 240 is half of 480, so 50% makes sense.	
Write a complete sentence that answers the question.	Of the total calories in each brownie, 50% is fat.

Note:
Exercise:

Problem:

Veronica is planning to make muffins from a mix. The package says each muffin will be 230 calories and 60 calories will be from fat. What percent of the total calories is from fat? (Round to the nearest whole percent.)

Solution:

26%

Note:
Exercise:

Problem:

The brownie mix Ricardo plans to use says that each brownie will be 190 calories, and 70 calories are from fat. What percent of the total calories are from fat?

Solution:

37%

Find Percent Increase and Percent Decrease

People in the media often talk about how much an amount has increased or decreased over a certain period of time. They usually express this increase or decrease as a percent.

To find the **percent increase**, first we find the amount of increase, which is the difference between the new amount and the original amount. Then we find what percent the amount of increase is of the original amount.

Note:
Find Percent Increase.
Step 1. Find the amount of increase.

- $\text{increase} = \text{new amount} - \text{original amount}$

Step 2. Find the percent increase as a percent of the original amount.

Example:
Exercise:

Problem:

In 2011, the California governor proposed raising community college fees from $26 per unit to $36 per unit. Find the percent increase. (Round to the nearest tenth of a percent.)

Solution:
Solution

What are you asked to find?	the percent increase
Choose a variable to represent it.	Let p = percent.
Find the amount of increase.	$\underbrace{36}_{\text{new amount}} - \underbrace{26}_{\text{original amount}} = \underbrace{10}_{\text{increase}}$
Find the percent increase.	The increase is what percent of the original amount?
Translate to an equation.	$\underbrace{10}_{10}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{what percent}}_{p}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{26?}_{26}$
Divide both sides by 26.	$\frac{10}{26} = \frac{26p}{26}$
Round to the nearest thousandth.	$0.384 = p$
Convert to percent form.	$38.4\% = p$
Write a complete sentence.	The new fees represent a 38.4% increase over the old fees.

Note:
Exercise:

Problem:

In 2011, the IRS increased the deductible mileage cost to 55.5 cents from 51 cents. Find the percent increase. (Round to the nearest tenth of a percent.)

Solution:

8.8%

Note:
Exercise:

Problem:

In 1995, the standard bus fare in Chicago was $\$1.50$. In 2008, the standard bus fare was $\$2.25$. Find the percent increase. (Round to the nearest tenth of a percent.)

Solution:

50%

Finding the **percent decrease** is very similar to finding the percent increase, but now the amount of decrease is the difference between the original amount and the final amount. Then we find what percent the amount of decrease is of the original amount.

Note:

Find percent decrease.

Find the amount of decrease.

- decrease $=$ original amount $-$ new amount

Find the percent decrease as a percent of the original amount.

Example:
Exercise:

Problem:

The average price of a gallon of gas in one city in June 2014 was $3.71. The average price in that city in July was $3.64. Find the percent decrease.

Solution:
Solution

What are you asked to find?	the percent decrease
Choose a variable to represent it.	Let $p =$ percent.
Find the amount of decrease.	$\underbrace{3.71}_{\text{original amount}} - \underbrace{3.64}_{\text{new amount}} = \underbrace{0.07}_{\text{increase}}$

Find the percent of decrease.	The decrease is what percent of the original amount?
Translate to an equation.	$\underbrace{0.07}_{0.07}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{what percent}}_{p}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{3.71}_{3.71}$
Divide both sides by 3.71.	$\frac{0.07}{3.71} = \frac{3.71p}{3.71}$
Round to the nearest thousandth.	$0.019 = p$
Convert to percent form.	$1.9\% = p$
Write a complete sentence.	The price of gas decreased 1.9%.

Note:
Exercise:

Problem:

The population of one city was about 672,000 in 2010. The population of the city is projected to be about 630,000 in 2020. Find the percent decrease. (Round to the nearest tenth of a percent.)

Solution:

6.3%

Note:
Exercise:

Problem:

Last year Sheila's salary was $\$42,000$. Because of furlough days, this year her salary was $\$37,800$. Find the percent decrease. (Round to the nearest tenth of a percent.)

Solution:

10%

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Percent Increase and Percent Decrease Visualization

Key Concepts

- **Solve an application.**

  Identify what you are asked to find and choose a variable to represent it.
  Write a sentence that gives the information to find it.
  Translate the sentence into an equation.
  Solve the equation using good algebra techniques.
  Write a complete sentence that answers the question.
  Check the answer in the problem and make sure it makes sense.

- **Find percent increase.**

  Find the amount of increase: $\text{increase} = \text{new amount} - \text{original amount}$

  Find the percent increase as a percent of the original amount.

- **Find percent decrease.**

  Find the amount of decrease. $\text{decrease} = \text{original amount} - \text{new amount}$

  Find the percent decrease as a percent of the original amount.

Practice Makes Perfect

Translate and Solve Basic Percent Equations

In the following exercises, translate and solve.

Exercise:

Problem: What number is 45% of 120?

Solution:

54

Exercise:

Problem: What number is 65% of 100?

Exercise:

Problem: What number is 24% of 112?

Solution:

26.88

Exercise:

Problem: What number is 36% of 124?

Exercise:

Problem: 250% of 65 is what number?

Solution:

162.5

Exercise:

Problem: 150% of 90 is what number?

Exercise:

Problem: 800% of 2,250 is what number?

Solution:

18,000

Exercise:

Problem: 600% of 1,740 is what number?

Exercise:

Problem: 28 is 25% of what number?

Solution:

112

Exercise:

Problem: 36 is 25% of what number?

Exercise:

Problem: 81 is 75% of what number?

Solution:

108

Exercise:

Problem: 93 is 75% of what number?

Exercise:

Problem: 8.2% of what number is \$2.87?

Solution:

\$35

Exercise:

Problem: 6.4% of what number is \$2.88?

Exercise:

Problem: 11.5% of what number is \$108.10?

Solution:

\$940

Exercise:

Problem: 12.3% of what number is \$92.25?

Exercise:

Problem: What percent of 260 is 78?

Solution:

30%

Exercise:

Problem: What percent of 215 is 86?

Exercise:

Problem: What percent of 1,500 is 540?

Solution:

36%

Exercise:

Problem: What percent of 1,800 is 846?

Exercise:

Problem: 30 is what percent of 20?

Solution:

150%

Exercise:

Problem: 50 is what percent of 40?

Exercise:

Problem: 840 is what percent of 480?

Solution:

175%

Exercise:

Problem: 790 is what percent of 395?

Solve Applications of Percents

In the following exercises, solve the applications of percents.

Exercise:

Problem:

Geneva treated her parents to dinner at their favorite restaurant. The bill was $\$74.25$. She wants to leave 16% of the total bill as a tip. How much should the tip be?

Solution:

$11.88

Exercise:

Problem:

When Hiro and his co-workers had lunch at a restaurant the bill was $\$90.50$. They want to leave 18% of the total bill as a tip. How much should the tip be?

Exercise:

Problem:

Trong has 12% of each paycheck automatically deposited to his savings account. His last paycheck was $\$2{,}165$. How much money was deposited to Trong's savings account?

Solution:

$259.80

Exercise:

Problem:

Cherise deposits 8% of each paycheck into her retirement account. Her last paycheck was $\$1,485$. How much did Cherise deposit into her retirement account?

Exercise:

Problem:

One serving of oatmeal has 8 grams of fiber, which is 33% of the recommended daily amount. What is the total recommended daily amount of fiber?

Solution:

24.2 grams

Exercise:

Problem:

One serving of trail mix has 67 grams of carbohydrates, which is 22% of the recommended daily amount. What is the total recommended daily amount of carbohydrates?

Exercise:

Problem:

A bacon cheeseburger at a popular fast food restaurant contains $2,070$ milligrams (mg) of sodium, which is 86% of the recommended daily amount. What is the total recommended daily amount of sodium?

Solution:

2,407 grams

Exercise:

Problem:

A grilled chicken salad at a popular fast food restaurant contains 650 milligrams (mg) of sodium, which is 27% of the recommended daily amount. What is the total recommended daily amount of sodium?

Exercise:

Problem:

The nutrition fact sheet at a fast food restaurant says the fish sandwich has 380 calories, and 171 calories are from fat. What percent of the total calories is from fat?

Solution:

45%

Exercise:

Problem:

The nutrition fact sheet at a fast food restaurant says a small portion of chicken nuggets has 190 calories, and 114 calories are from fat. What percent of the total calories is from fat?

Exercise:

Problem:

Emma gets paid \$3,000 per month. She pays \$750 a month for rent. What percent of her monthly pay goes to rent?

Solution:

25%

Exercise:

Problem:

Dimple gets paid \$3,200 per month. She pays \$960 a month for rent. What percent of her monthly pay goes to rent?

Find Percent Increase and Percent Decrease

In the following exercises, find the percent increase or percent decrease.

Exercise:

Problem:

Tamanika got a raise in her hourly pay, from \$15.50 to \$17.55. Find the percent increase.

Solution:

13.2%

Exercise:

Problem:

Ayodele got a raise in her hourly pay, from \$24.50 to \$25.48. Find the percent increase.

Exercise:

Problem:

Annual student fees at the University of California rose from about \$4,000 in 2000 to about \$9,000 in 2014. Find the percent increase.

Solution:

125%

Exercise:

Problem:

The price of a share of one stock rose from $12.50 to $50. Find the percent increase.

Exercise:

Problem:

According to Time magazine (7/19/2011) annual global seafood consumption rose from 22 pounds per person in 1960 to 38 pounds per person today. Find the percent increase. (Round to the nearest tenth of a percent.)

Solution:

72.7%

Exercise:

Problem:

In one month, the median home price in the Northeast rose from $225,400 to $241,500. Find the percent increase. (Round to the nearest tenth of a percent.)

Exercise:

Problem:

A grocery store reduced the price of a loaf of bread from $2.80 to $2.73. Find the percent decrease.

Solution:

2.5%

Exercise:

Problem:

The price of a share of one stock fell from $\$8.75$ to $\$8.54$. Find the percent decrease.

Exercise:

Problem:

Hernando's salary was $\$49,500$ last year. This year his salary was cut to $\$44,055$. Find the percent decrease.

Solution:

11%

Exercise:

Problem:

From 2000 to 2010, the population of Detroit fell from about $951,000$ to about $714,000$. Find the percent decrease. (Round to the nearest tenth of a percent.)

Exercise:

Problem:

In one month, the median home price in the West fell from $\$203,400$ to $\$192,300$. Find the percent decrease. (Round to the nearest tenth of a percent.)

Solution:

5.5%

Exercise:

Problem:

Sales of video games and consoles fell from $\$1,150$ million to $\$1,030$ million in one year. Find the percent decrease. (Round to the nearest tenth of a percent.)

Everyday Math

Exercise:

Problem:

Tipping At the campus coffee cart, a medium coffee costs $\$1.65$. MaryAnne brings $\$2.00$ with her when she buys a cup of coffee and leaves the change as a tip. What percent tip does she leave?

Solution:

21.2%

Exercise:

Problem:

Late Fees Alison was late paying her credit card bill of $\$249$. She was charged a 5% late fee. What was the amount of the late fee?

Writing Exercises

Exercise:

Problem:

Without solving the problem $\text{\textquotedblleft}44$ is 80% of what number”, think about what the solution might be. Should it be a number that is greater than 44 or less than 44? Explain your reasoning.

Solution:

The original number should be greater than 44.80% is less than 100%, so when 80% is converted to a decimal and multiplied to the base in the percent equation, the resulting amount of 44 is less. 44 is only the larger number in cases where the percent is greater than 100%.

Exercise:

Problem:

Without solving the problem “What is 20% of 300 ?” think about what the solution might be. Should it be a number that is greater than 300 or less than 300? Explain your reasoning.

Exercise:

Problem:

After returning from vacation, Alex said he should have packed 50% fewer shorts and 200% more shirts. Explain what Alex meant.

Solution:

Alex should have packed half as many shorts and twice as many shirts.

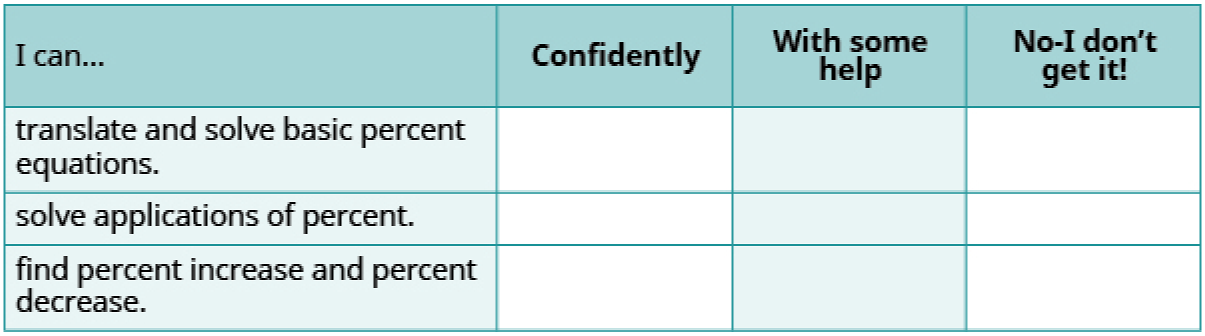
Exercise:

Problem:

Because of road construction in one city, commuters were advised to plan their Monday morning commute to take 150% of their usual commuting time. Explain what this means.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
translate and solve basic percent equations.			
solve applications of percent.			
find percent increase and percent decrease.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

percent increase

The percent increase is the percent the amount of increase is of the original amount.

percent decrease

The percent decrease is the percent the amount of decrease is of the original amount.

A Review of Percents

Learning Objectives

By the end of this lesson, you will be able to:

- Understand the meaning of percent
- Convert a fraction to a percent
- Convert a percent to a fraction
- Convert a decimal to a percent
- Convert a percent to a decimal

The Meaning of Percent

The word **percent** comes from the Latin word “per centum,” “per” meaning “for each,” and “centum” meaning “hundred.”

Percent (%)
Percent means "per hundred", or “for each hundred”, or “for every hundred.” The symbol % is used to represent the word percent.

Thus, $1\% = \frac{1}{100}$ or $1\% = 0.01.$

Converting A Fraction To A Percent

We can see how a fraction can be converted to a percent by analyzing the method that $\frac{3}{5}$ is converted to a percent. In order to convert $\frac{3}{5}$ to a percent, we need to introduce $\frac{1}{100}$ (since percent means for each hundred).

Example:

$$\begin{aligned}
\frac{3}{5} &= \frac{3}{5}\cdot\frac{100}{100} && \text{Multiply the fraction by 1.}\\
&= \frac{3}{5}\cdot 100\cdot\frac{1}{100} && \text{Since } \frac{100}{100} = 100\cdot\frac{1}{100}.\\
&= \frac{300}{5}\cdot\frac{1}{100} && \text{Divide 300 by 5.}\\
&= 60\cdot\frac{1}{100} && \text{Multiply the fractions.}\\
&= 60\% && \text{Replace } \frac{1}{100} \text{ with the } \% \text{ symbol.}
\end{aligned}$$

Fraction to Percent

To convert a fraction to a percent, multiply the fraction by 1 in the form $100\cdot\frac{1}{100}$, then replace $\frac{1}{100}$ with the % symbol.

In the next three examples, we will convert each fraction to a percent.

Example:

$$\begin{aligned}
\frac{1}{4} &= \frac{1}{4}\cdot 100\cdot\frac{1}{100}\\
&= \frac{100}{4}\cdot\frac{1}{100}\\
&= 25\cdot\frac{1}{100}\\
&= 25\%
\end{aligned}$$

Example:

$$\begin{aligned}
\frac{8}{5} &= \frac{8}{5}\cdot 100\cdot\frac{1}{100}\\
&= \frac{800}{5}\cdot\frac{1}{100}\\
&= 160\%
\end{aligned}$$

Example:

$$\begin{aligned}
\frac{4}{9} &= \frac{4}{9} \cdot 100 \cdot \frac{1}{100} \\
&= \frac{400}{9} \cdot \frac{1}{100} \\
&= (44.4...) \cdot \frac{1}{100} \\
&= \left(44.4\right) \cdot \frac{1}{100} \\
&= 44.4\%
\end{aligned}$$

Note:
Try It
Exercise:

Problem: Convert to a percent: $\frac{3}{5}$.

Solution:

37.5%

Converting a Percent to a Fraction

Note:
Convert a percent to a fraction.

Write the percent as a ratio with the denominator 100.
Simplify the fraction if possible.

Example:
Exercise:

Problem: Convert each percent to a fraction:

ⓐ 36%
ⓑ 125%

Solution:
Solution

ⓐ	
	36%
Write as a ratio with denominator 100.	$\frac{36}{100}$
Simplify.	$\frac{9}{25}$

ⓑ	
	125%
Write as a ratio with denominator 100.	$\frac{125}{100}$

Simplify.	$\frac{5}{4}$

Note:
Try It
Exercise:

Problem: Convert each percent to a fraction:

ⓐ 48%
ⓑ 110%

Solution:

ⓐ $\frac{12}{25}$
ⓑ $\frac{11}{10}$

Note:
Try It
Exercise:

Problem: Convert each percent to a fraction:

ⓐ 64%
ⓑ 150%

Solution:

ⓐ $\frac{16}{25}$

ⓑ $\frac{3}{2}$

The previous example shows that a percent can be greater than 1. We saw that 125% means $\frac{125}{100}$, or $\frac{5}{4}$. These are improper fractions, and their values are greater than one.

Example:
Exercise:

Problem: Convert each percent to a fraction:

ⓐ 24.5%
ⓑ $33\frac{1}{3}\%$

Solution:
Solution

ⓐ	
	24.5%
Write as a ratio with denominator 100.	$\frac{24.5}{100}$
Clear the decimal by multiplying numerator and denominator by 10.	$\frac{24.5(10)}{100(10)}$

Multiply.	$\frac{245}{1000}$
Rewrite showing common factors.	$\frac{5 \cdot 49}{5 \cdot 200}$
Simplify.	$\frac{49}{200}$

ⓑ	
	$33\frac{1}{3}\%$
Write as a ratio with denominator 100.	$\frac{33\frac{1}{3}}{100}$
Write the numerator as an improper fraction.	$\frac{\frac{100}{3}}{100}$
Rewrite as fraction division, replacing 100 with $\frac{100}{1}$.	$\frac{100}{3} \div \frac{100}{1}$
Multiply by the reciprocal.	$\frac{100}{3} \cdot \frac{1}{100}$
Simplify.	$\frac{1}{3}$

Note:
Try It
Exercise:

Problem: Convert each percent to a fraction:

ⓐ 64.4%
ⓑ $66\frac{2}{3}\%$

Solution:

ⓐ $\frac{161}{250}$
ⓑ $\frac{2}{3}$

Note:
Try It
Exercise:

Problem: Convert each percent to a fraction:

ⓐ 42.5%
ⓑ $8\frac{3}{4}\%$

Solution:

ⓐ $\frac{113}{250}$
ⓑ $\frac{7}{80}$

Converting a Decimal to a Percent

We can see how a decimal is converted to a percent by analyzing the method that 0.75 is converted to a percent. We need to introduce $\frac{1}{100}$.

$$\begin{aligned} 0.75 &= 0.75 \cdot 100 \cdot \frac{1}{100} && \text{Multiply the decimal by 1.} \\ &= 75 \cdot \frac{1}{100} \\ &= 75\% && \text{Replace } \frac{1}{100} \text{ with the } \% \text{ symbol.} \end{aligned}$$

Decimal to Percent
To convert a fraction to a percent, multiply the decimal by 1 in the form $100 \cdot \frac{1}{100}$, then replace $\frac{1}{100}$ with the % symbol. This amounts to moving the decimal point 2 places to the right.

In the next three examples, we will convert each decimal to a percent.

Example:

$$\begin{aligned} 0.62 &= 0.62 \cdot 100 \cdot \frac{1}{100} \\ &= 62 \cdot \frac{1}{100} \\ &= 62\% \end{aligned}$$

Notice that the decimal point in the original number has been moved to the right 2 places.

Example:

$$\begin{aligned} 8.4 &= 8.4 \cdot 100 \cdot \frac{1}{100} \\ &= 840 \cdot \frac{1}{100} \\ &= 840\% \end{aligned}$$

Notice that the decimal point in the original number has been moved to the right 2 places.

Example:

$$\begin{aligned} 0.47623 &= 0.47623 \cdot 100 \cdot \tfrac{1}{100} \\ &= 0.47623 \cdot \tfrac{1}{100} \\ &= 47.623\% \end{aligned}$$

Notice that the decimal point in the original number has been moved to the right 2 places.

Note:
Try It
Exercise:

Problem: Convert to a percent: 0.0362.

Solution:

3.62%

Converting A Percent To A Decimal

We can see how a percent is converted to a decimal by analyzing the method that 12% is converted to a decimal. We need to introduce $\frac{1}{100}$.

$$\begin{aligned} 12\% &= 12 \cdot \tfrac{1}{100} && \text{Replace } \% \text{ with } \tfrac{1}{100}. \\ &= \tfrac{12}{100} && \text{Multiply the fractions.} \\ &= 0.12 && \text{Divide 12 by 100.} \end{aligned}$$

Percent to Decimal

To convert a percent to a decimal, replace the % symbol with $\frac{1}{100}$, then divide the number by 100. This amounts to moving the decimal point 2 places to the left.

In the next three examples, we will convert each percent to a decimal.

Example:

$$\begin{aligned} 48\% &= 48 \cdot \tfrac{1}{100} \\ &= \tfrac{48}{100} \\ &= 0.48 \end{aligned}$$

Notice that the decimal point in the original number has been moved to the left 2 places.

Example:

$$\begin{aligned} 659\% &= 659 \cdot \tfrac{1}{100} \\ &= \tfrac{659}{100} \\ &= 6.59 \end{aligned}$$

Notice that the decimal point in the original number has been moved to the left 2 places.

Example:

$$\begin{aligned} 0.4113\% &= 0.4113 \cdot \frac{1}{100} \\ &= \frac{0.4113}{100} \\ &= 0.004113 \end{aligned}$$

Notice that the decimal point in the original number has been moved to the left 2 places.

Note:
Try It
Exercise:

Problem: Convert to a decimal: 4.7%.

Solution:

0.047

Homework

For the following problems, convert each fraction to a percent.
Exercise:

Problem: $\frac{2}{5}$

Solution:

40%

Exercise:

Problem: $\frac{7}{8}$

Exercise:

Problem: $\frac{1}{8}$

Solution:

12.5%

Exercise:

Problem: $\frac{5}{16}$

Exercise:

Problem: $15 \div 22$

Solution:

68.18%

Exercise:

Problem: $\frac{2}{11}$

Exercise:

Problem: $\frac{2}{9}$

Solution:

22.22%

Exercise:

Problem: $\frac{16}{45}$

Exercise:

Problem: $\frac{27}{55}$

Solution:

49.09%

Exercise:

Problem: $\frac{7}{27}$

Exercise:

Problem: 15

Solution:

1500%

Exercise:

Problem: 8

In the following exercises, convert each percent to a fraction and simplify all fractions.

Exercise:

Problem: 4%

Solution:

$\frac{1}{25}$

Exercise:

Problem: 8%

Exercise:

Problem: 17%

Solution:

$\frac{17}{100}$

Exercise:

Problem: 19%

Exercise:

Problem: 52%

Solution:

$\frac{13}{25}$

Exercise:

Problem: 78%

Exercise:

Problem: 125%

Solution:

$\frac{5}{4}$

Exercise:

Problem: 135%

Exercise:

Problem: 37.5%

Solution:

$\frac{3}{8}$

Exercise:

Problem: 42.5%

Exercise:

Problem: 18.4%

Solution:

$\frac{23}{125}$

Exercise:

Problem: 46.4%

Exercise:

Problem: $9\frac{1}{2}\%$

Solution:

$\frac{19}{200}$

Exercise:

Problem: $8\frac{1}{2}\%$

Exercise:

Problem: $5\frac{1}{3}\%$

Solution:

$\frac{4}{75}$

Exercise:

Problem: $6\frac{2}{3}\%$

For the following problems, convert each decimal to a percent.

Exercise:

Problem:0.36

Solution:

36%

Exercise:

Problem:0.42

Exercise:

Problem:0.446

Solution:

44.6%

Exercise:

Problem:0.1298

Exercise:

Problem:4.25

Solution:

425%

Exercise:

Problem:5.875

Exercise:

Problem:86.98

Solution:

8698%

Exercise:

Problem:21.26

Exercise:

Problem:14

Solution:

1400%

Exercise:

Problem:12

For the following problems, convert each percent to a decimal.

Exercise:

Problem:35%

Solution:

0.35

Exercise:

Problem:76%

Exercise:

Problem:18.6%

Solution:

0.186

Exercise:

Problem:67.2%

Exercise:

Problem:9.0145%

Solution:

0.090145

Exercise:

Problem:3.00156%

Exercise:

Problem:0.00005%

Solution:

0.0000005

Exercise:

Problem:0.00034%

Scientific notation

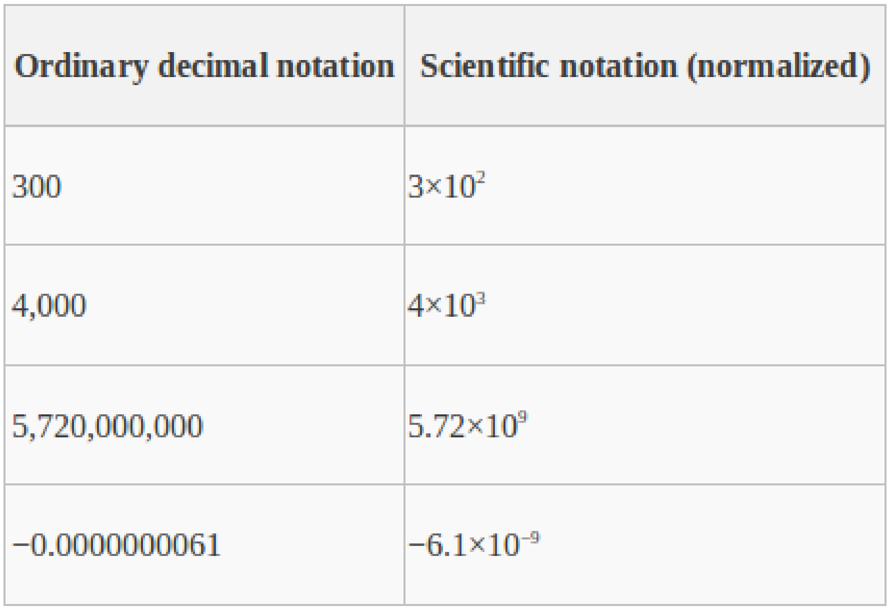
Scientific Notation

Scientific notation, also known as **standard form** or as **exponential notation**, is a way of writing numbers that accommodates values too large or small to be conveniently written in standard decimal notation. Scientific notation has a number of useful properties and is often favored by scientists, mathematicians, and engineers, who work with such numbers.

In scientific notation all numbers are written like this:
Equation:

$$a \times 10^b$$

(" *a* times ten to the power of *b* "), where the **exponent** *b* is an **integer** , and the **coefficient** *a* is any **real number** (but see normalized notation below), called the *significand* or *mantissa* (though the term "mantissa" may cause confusion as it can also refer to the **fractional** part of the common **logarithm**). If the number is negative then a minus sign precedes *a* (as in ordinary decimal notation).



Ordinary decimal notation	Scientific notation (normalized)
300	3×10^{2}
4,000	4×10^{3}
5,720,000,000	5.72×10^{9}
−0.0000000061	-6.1×10^{-9}

What is the use of scientific notation in every day life?

Scientific notation is needed any time you need to express a number that is very big or very small. Suppose for example you wanted to figure out how many drops of water were in a river 12 km long, 270 m wide, and 38 m deep (assuming one drop is one milliliter). It's much more compact and meaningful to write the answer as roughly 1.23×10^{-14} than it is to write 123120000000000. For one thing, the scientific notation is easier to read, and makes it much easier to tell at a glance what the order of magnitude is (rather than counting zeros).

For another, most of the digits in 123120000000000 are completely meaningless (unless your measurements were very precise). For instance, if the exact river length were really 12.123123 km (we just measured it to the nearest kilometers), then correct number of drops would be 124383242000000, and after the first three digits our result of 123120000000000 is quite inaccurate. So it's better to use a notation (like scientific notation) in which you can **suppress the inaccurate digits**.

Converting numbers to scientific notation

Count the number of places to move the decimal point to the right or left and write the number like this:

The number 1,350,000,000 can be written as
Equation:

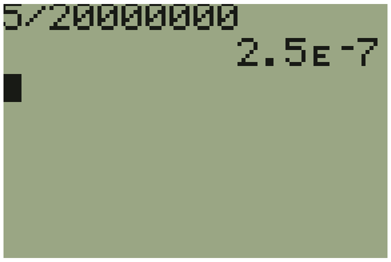
$$1.35 \times 10^{9}$$

The number 0.000000000000017 can be written as
Equation:

$$1.7 \times 10^{-14}$$

Calculator Representation of Scientific Numbers

If we wanted to divide 5 by 20,000,000, the calculator will show



```
5/20000000
             2.5E-7
```

On this screen, the E in $2.5E{-}7$ stands for 10^{-7}. The number $2.5E^{-7}$ is equivalent to 2.5×10^{-7}.

Additional Resources

- **Youtube**: TI Calculator Tutorial: Scientific Notation
- **StackExchange**: What does E mean in 9.0122222900391E-5?

Factorials
Introduction to factorials

Factorials

Given a positive number n, the factorial of n is written as $n!$. This notation is a shorthand for writing the product of all positive numbers less than or equal to n.

For example:
$3! = 3 \times 2 \times 1$

$4! = 4 \times 3 \times 2 \times 1$

Note: By definition, $0! = 1$

Further Reading

For detailed discussion of factorials, please visit Mathisfun.

Geometric Mean
Calculate geometric means

Geometric Mean

Geometric mean is one of the ways to measure the center or the typical value of a data set. If we have two numbers a and b, their geometric mean is simply the square root of their product.

Geometric mean of and = .

Further Reading

For detailed explanation, please visit:

- **Mathisfun**: Geometric Mean
- **UofToronto**: Arithmetic and Geometric Sequences
- **PurpleMath**: Arithmetic & Geometric Sequences

Functions and Function Notation
In this section, you will:

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

A **relation** is a set of ordered pairs. The set consisting of the first components of each ordered pair is called the **domain** and the set consisting of the second components of each ordered pair is called the **range**. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.
Equation:

$$\{(1,\ 2),\ (2,\ 4),\ (3,\ 6),\ (4,\ 8),\ (5,\ 10)\}$$

The domain is $\{1,\ 2,\ 3,\ 4,\ 5\}$. The range is $\{2,\ 4,\ 6,\ 8,\ 10\}$.

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an **output** value, or **dependent variable**, and is often labeled lowercase letter y.

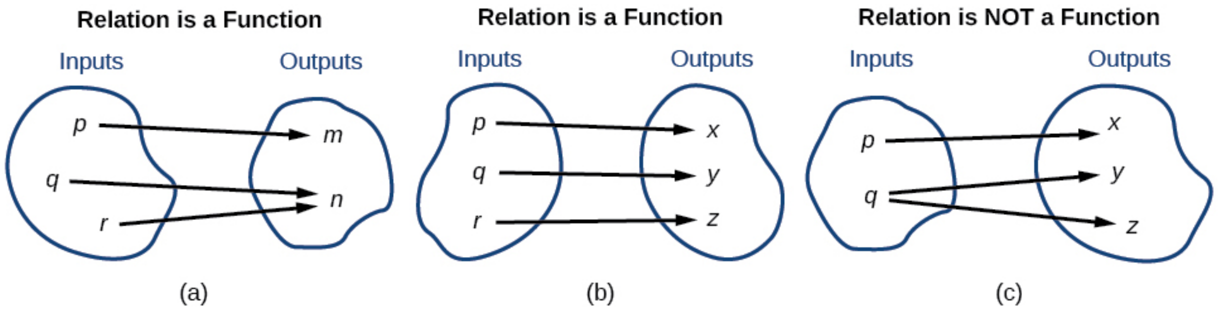
A function f is a relation that assigns a single element in the range to each element in the domain. In other words, no x-values are repeated. For our example that relates the first five natural numbers to numbers double their values, this relation is a function because each element in the domain, $\{1,\ 2,\ 3,\ 4,\ 5\}$, is paired with exactly one element in the range, $\{2,\ 4,\ 6,\ 8,\ 10\}$.

Now let’s consider the set of ordered pairs that relates the terms “even” and “odd” to the first five natural numbers. It would appear as
Equation:

$$\{(\text{odd},\ 1),\ (\text{even},\ 2),\ (\text{odd},\ 3),\ (\text{even},\ 4),\ (\text{odd},\ 5)\}$$

Notice that each element in the domain, $\{\text{even}, \text{odd}\}$ is *not* paired with exactly one element in the range, $\{1,\ 2,\ 3,\ 4,\ 5\}$. For example, the term “odd” corresponds to three values from the domain, $\{1,\ 3,\ 5\}$ and the term “even” corresponds to two values from the range, $\{2,\ 4\}$. This violates the definition of a function, so this relation is not a function.

[link] compares relations that are functions and not functions.


Relation is a Function
Inputs
Outputs
p
q
r
m
n
(a)
Relation is a Function
Inputs
Outputs
p
q
r
x
y
z
(b)
Relation is NOT a Function
Inputs
Outputs
p
q
x
y
z
(c)

(a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output $n.$ (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.

Note:
Function
A **function** is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."
The **input** values make up the **domain**, and the **output** values make up the **range**.

Note:
Given a relationship between two quantities, determine whether the relationship is a function.

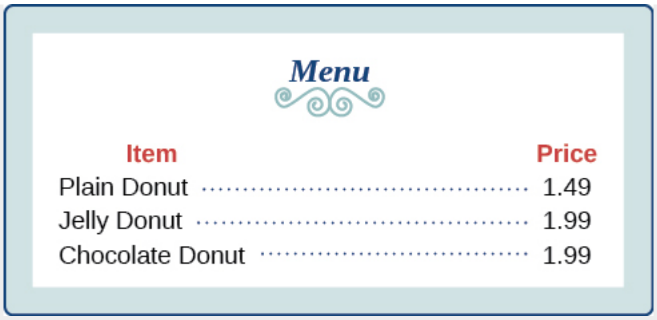
1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Example:
Exercise:

Problem:
Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in [link] consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?

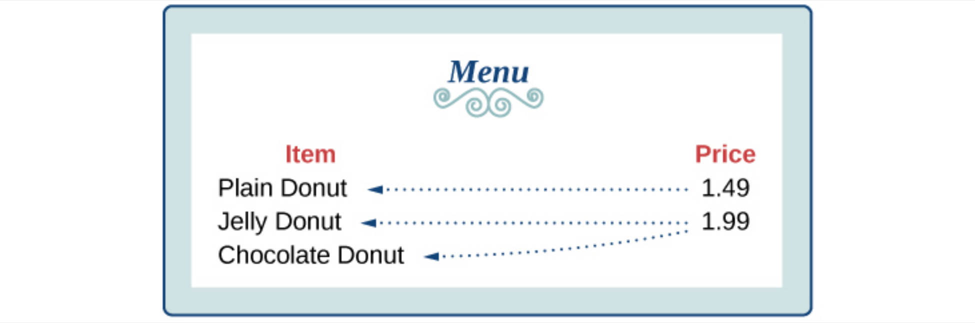

Menu
Item
Price
Plain Donut
1.49
Jelly Donut
1.99
Chocolate Donut
1.99

Solution:

a. Let’s begin by considering the input as the items on the menu. The output values are then the prices.

   Each item on the menu has only one price, so the price is a function of the item.

b. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See [link].


Menu
Item
Price
Plain Donut
1.49
Jelly Donut
1.99
Chocolate Donut

   Therefore, the item is a not a function of price.

Example:
Exercise:

Problem:
Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade-point average. Is grade-point average a function of the percent grade? Is the percent grade a function of the grade-point average? [link] shows a possible rule for assigning grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade-point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution:

For any percent grade earned, there is an associated grade-point average, so the grade-point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade-point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade-point average.

Note:
Exercise:

Problem:[link][footnote] lists the five greatest baseball players of all time in order of rank.
http://www.baseball-almanac.com/legendary/lisn100.shtml. Accessed 3/24/2014.

Player	**Rank**
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Solution:

a. yes; b. yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into graphing calculators and computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent “height is a function of age,” we start by identifying the descriptive variables h for height and a for age. The letters $f,\ g,$and h are often used to represent functions just as we use $x,\ y,$and$z$ to represent numbers and

A, B, and C to represent sets.

Equation:

h is f of a	We name the function f; height is a function of age.
$h = f(a)$	We use parentheses to indicate the function input.
$f(a)$	We name the function f; the expression is read as “f of a.”

Remember, we can use any letter to name the function; the notation $h\left(a\right)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f\left(a+b\right)$ means “first add a and b, and the result is the input for the function f.” The operations must be performed in this order to obtain the correct result.

Note:
Function Notation
The notation $y=f\left(x\right)$ defines a function named f. This is read as “y is a function of x.” The letter x represents the input value, or independent variable. The letter y, or $f\left(x\right)$, represents the output value, or dependent variable.

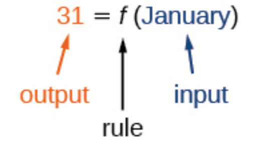
Example:
Exercise:

Problem:
Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month. Assume that the domain does not include leap years.

Solution:

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days} = f(\text{month})$ or $d = f(m)$. The name of the month is the input to a “rule” that associates a specific number (the output) with each input.



For example, $f\left(\text{March}\right)=31$, because March has 31 days. The notation $d=f\left(m\right)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example:

Exercise:

Problem:
Interpreting Function Notation

A function $N = f\left(y \right)$ gives the number of police officers, $N,$ in a town in year $y.$ What does $f\left(2005 \right) = 300$ represent?

Solution:

When we read $f\left(2005 \right) = 300,$ we see that the input year is 2005. The value for the output, the number of police officers $\left(N \right),$ is 300. Remember, $N = f\left(y \right).$ The statement $f\left(2005 \right) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Note:
Exercise:

Problem: Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Solution:

$w = f(d)$

Note:
Instead of a notation such as $y = f(x\ ),$ could we use the same symbol for the output as for the function, such as $y = y(x\ ),$ meaning "y is a function of x?"
Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f\left(x \right), P = W\left(d \right),$ and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

[link] lists the input number of each month (January = 1, February = 2, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f\left(m \right)$ identifies months by an integer rather than by name.

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12

Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

[link] defines a function $Q = g(n)$. Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q .

n	1	2	3	4	5
Q	8	6	7	6	8

[link] displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

Note:
Given a table of input and output values, determine whether the table represents a function.

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

Example:
Exercise:

Problem:
Identifying Tables that Represent Functions

Which table, [link], [link], or [link], represents a function (if any)?

Input	**Output**
2	1

Input	Output
5	3
8	6

Input	Output
–3	5
0	1
4	5

Input	Output
1	0
5	2
5	4

Solution:

[link] and [link] define functions. In both, each input value corresponds to exactly one output value. [link] does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by [link] can be represented by writing
Equation:

$$f(2) = 1, f(5) = 3, \text{and } f(8) = 6$$

Similarly, the statements
Equation:

$$g\left(-3\right) = 5,\ g\left(0\right) = 1, \text{and } g\left(4\right) = 5$$

represent the function in [link].

[link] cannot be expressed in a similar way because it does not represent a function.

Note:
Exercise:

Problem: Does [link] represent a function?

Input	Output
1	10
2	100
3	1000

Solution:

yes

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we *evaluate* the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function’s formula and *solve* for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f\left(x\right)=5-3{x}^{2}$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

Note:
Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example:
Exercise:

Problem:
Evaluating Functions at Specific Values

Evaluate $f\left(x\right)=x^2+3x-4$ at

a. 2
b. a
c. $a+h$
d. $\frac{f(a+h)-f(a)}{h}$

Solution:

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.
Equation:

$$\begin{array}{ccc}f(2) & = & 2^2+3(2)-4\\ & = & 4+6-4\\ & = & 6\end{array}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.
Equation:

$$f\left(a\right)=a^2+3a-4$$

c. With an input value of $a+h,$ we must use the distributive property.
Equation:

$$\begin{array}{ccc}f(a+h) & = & (a+h)^2+3(a+h)-4\\ & = & a^2+2ah+h^2+3a+3h-4\end{array}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that
Equation:

$$f\left(a+h\right)=a^2+2ah+h^2+3a+3h-4$$

and we know that
Equation:

$$f\left(a\right)=a^2+3a-4$$

Now we combine the results and simplify.
Equation:

$$\begin{array}{rcll} \frac{f(a+h)-f(a)}{h} & = & \frac{\left(a^2+2ah+h^2+3a+3h-4\right)-\left(a^2+3a-4\right)}{h} & \\ & = & \frac{2ah+h^2+3h}{h} & \\ & = & \frac{h(2a+h+3)}{h} & \text{Factor out } h. \\ & = & 2a+h+3 & \text{Simplify.} \end{array}$$

Example:
Exercise:

Problem:
Evaluating Functions

Given the function $h\left(p\right)=p^2+2p,$ evaluate $h\left(4\right).$

Solution:

To evaluate $h\left(4\right),$ we substitute the value 4 for the input variable p in the given function.
Equation:

$$\begin{array}{rcl} h(p) & = & p^2+2p \\ h(4) & = & (4)^2+2(4) \\ & = & 16+8 \\ & = & 24 \end{array}$$

Therefore, for an input of 4, we have an output of 24.

Note:
Exercise:

Problem: Given the function $g\left(m\right)=\sqrt{m-4},$ evaluate $g\left(5\right).$

Solution:

$g\left(5\right)=1$

Example:
Exercise:

Problem:
Solving Functions

Given the function $h\left(p\right)=p^2+2p,$ solve for $h\left(p\right)=3.$

Solution:
Equation:

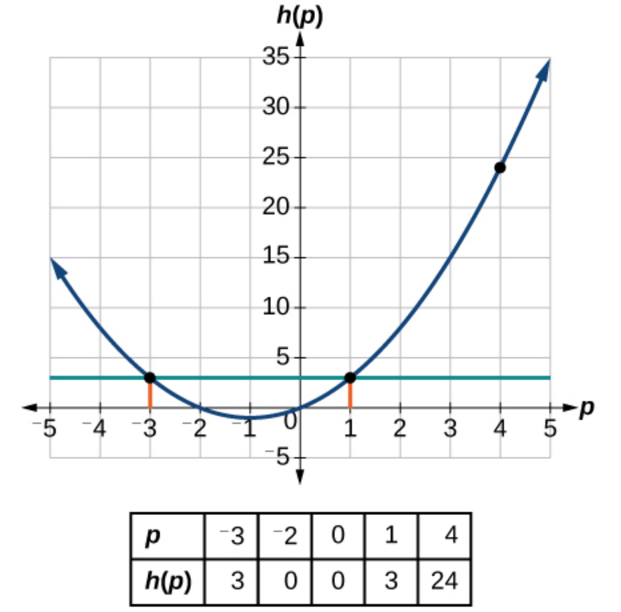
$$\begin{array}{rcll} h(p) & = & 3 & \\ p^2+2p & = & 3 & \text{Substitute the original function } h(p)=p^2+2p. \\ p^2+2p-3 & = & 0 & \text{Subtract 3 from each side.} \\ (p+3)(p-1) & = & 0 & \text{Factor.} \end{array}$$

If $(p+3)(p-1)=0$, either $(p+3)=0$ or $(p-1)=0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

Equation:

$$\begin{array}{rcllcl} (p+3) & = & 0, & p & = & -3 \\ (p-1) & = & 0, & p & = & 1 \end{array}$$

This gives us two solutions. The output $h(p)=3$ when the input is either $p=1$ or $p=-3$. We can also verify by graphing as in [link]. The graph verifies that $h(1)=h(-3)=3$ and $h(4)=24$.



p	−3	−2	0	1	4
h(p)	3	0	0	3	24

Note:
Exercise:

Problem: Given the function $g(m)=\sqrt{m-4}$, solve $g(m)=2$.

Solution:

$m=8$

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in equation form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n+6p=12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

Note:
Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves *only* the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

Example:
Exercise:

Problem:
Finding an Equation of a Function

Express the relationship $2n+6p=12$ as a function $p=f(n)$, if possible.

Solution:

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as $p=[\text{expression involving } n]$.
Equation:

$$\begin{array}{rcll} 2n+6p & = & 12 & \\ 6p & = & 12-2n & \text{Subtract } 2n \text{ from both sides.} \\ p & = & \frac{12-2n}{6} & \text{Divide both sides by 6 and simplify.} \\ p & = & \frac{12}{6}-\frac{2n}{6} & \\ p & = & 2-\frac{1}{3}n & \end{array}$$

Therefore, p as a function of n is written as
Equation:

$$p=f(n)=2-\frac{1}{3}n$$

Example:
Exercise:

Problem:
Expressing the Equation of a Circle as a Function

Does the equation $x^2+y^2=1$ represent a function with x as input and y as output? If so, express the relationship as a function $y=f(x)$.

Solution:

First we subtract x^2 from both sides.
Equation:

$$y^2 = 1 - x^2$$

We now try to solve for y in this equation.
Equation:

$$\begin{aligned} y &= \pm\sqrt{1-x^2} \\ &= +\sqrt{1-x^2} \text{ and } -\sqrt{1-x^2} \end{aligned}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$. If we graph both functions on a graphing calculator, we will get the upper and lower semicircles.

Note:
Exercise:

Problem: If $x - 8y^3 = 0,$ express y as a function of x.

Solution:

$y = f(x) = \frac{\sqrt[3]{x}}{2}$

Note:
Are there relationships expressed by an equation that do represent a function but that still cannot be represented by an algebraic formula?
Yes, this can happen. For example, given the equation $x = y + 2^y,$ *if we want to express* y *as a function of* x, *there is no simple algebraic formula involving only* x *that equals* $y.$ *However, each* x *does determine a unique value for* $y,$ *and there are mathematical procedures by which* y *can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for* y *as a function of* $x,$ *even though the formula cannot be written explicitly.*

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table. See [link].[footnote]
http://www.kgbanswers.com/how-long-is-a-dogs-memory-span/4221590. Accessed 3/24/2014.

Pet	Memory span in hours
Puppy	0.008
Adult dog	0.083
Cat	16
Goldfish	2160
Beta fish	3600

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet’s memory span lasts. We can evaluate the function P at the input value of “goldfish.” We would write $P(\text{goldfish}) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

Note:
Given a function represented by a table, identify specific output and input values.

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

Example:
Exercise:

Problem:
Evaluating and Solving a Tabular Function

Using [link],

a. Evaluate $g\left(3\right)$.
b. Solve $g\left(n\right) = 6$.

n	1	2	3	4	5
$g\left(n\right)$	8	6	7	6	8

Solution:

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n,that produce an output value of 6. [link] shows two solutions: 2 and 4.

n	1	**2**	3	**4**	5
$g(n)$	8	**6**	7	**6**	8

When we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

Note:
Exercise:

Problem: Using [link], evaluate $g(1)$.

Solution:

$g(1) = 8$

Finding Function Values from a Graph

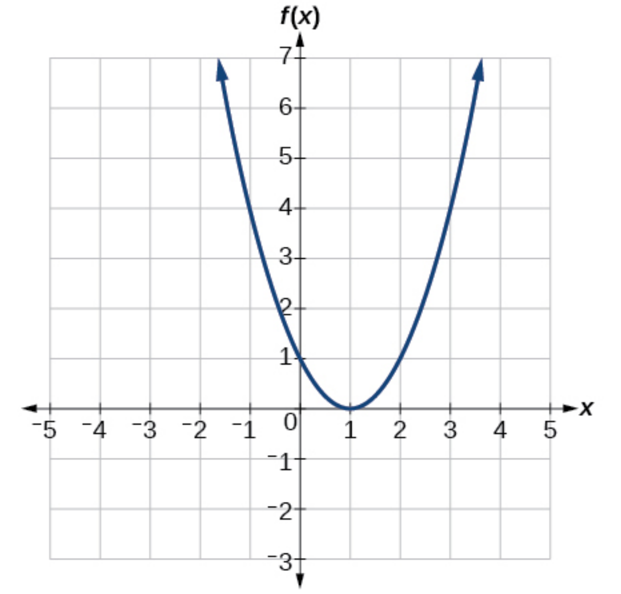
Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example:
Exercise:

Problem:
Reading Function Values from a Graph

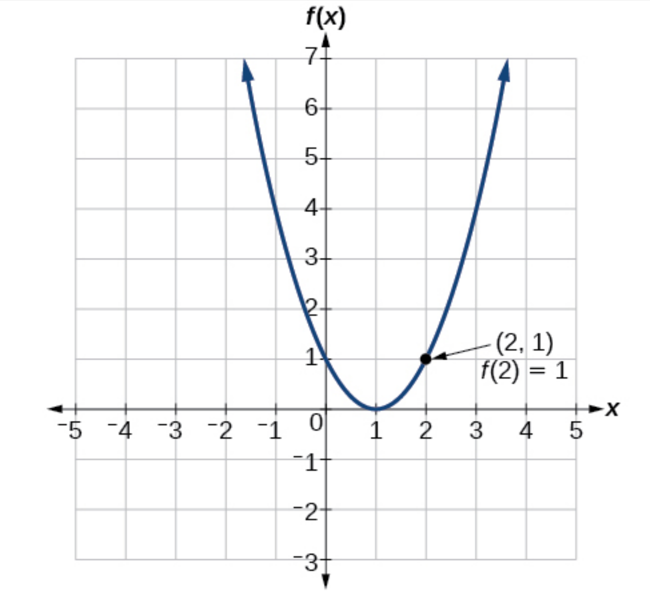
Given the graph in [link],

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

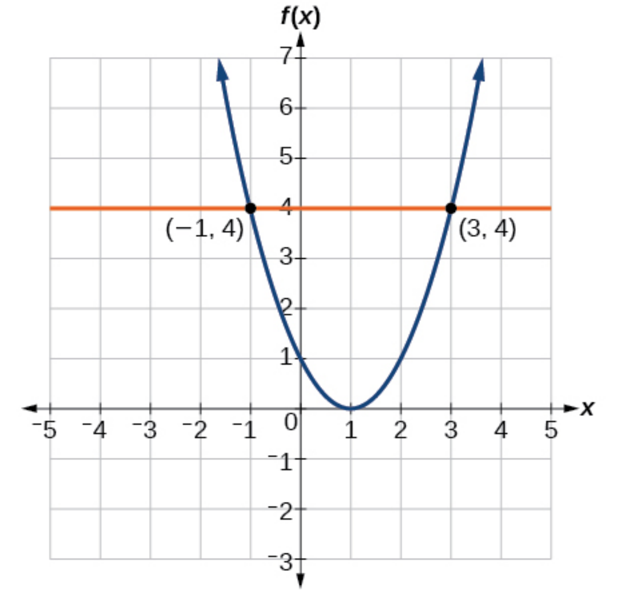


Solution:

a. To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See [link].



b. To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: -1 or 3. This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or3, the output is 4. See [link].



Note:
Exercise:

Problem: Using [link], solve $f\left(x\right)=1.$

Solution:

$x=0\text{ or }x=2$

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in [link] at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in [link].

Letter grade	Grade point average
A	4.0
B	3.0

Letter grade	Grade point average
C	2.0
D	1.0

This grading system represents a one-to-one function because each letter input yields one particular grade-point average output and each grade-point average corresponds to one input letter.

To visualize this concept, let’s look again at the two simple functions sketched in [link]**(a)** and [link]**(b)**. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output $n.$ The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

Note:
One-to-One Function
A **one-to-one function** is a function in which each output value corresponds to exactly one input value. There are no repeated *x*- or *y*-values.

Example:
Exercise:

Problem:
Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution:

A circle of radius r has a unique area measure given by $A=\pi r^2$,so for any input, $r,$ there is only one output, $A.$ The area is a function of radius $r.$

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A=\pi r^2.$ Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$.So the area of a circle is a one-to-one function of the circle’s radius.

Note:
Exercise:

Problem:

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Solution:

a. yes, because each bank account has a single balance at any given time; b. no, because several bank account numbers may have the same balance; c. no, because the same output may correspond to more than one input.

Note:
Exercise:

Problem:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

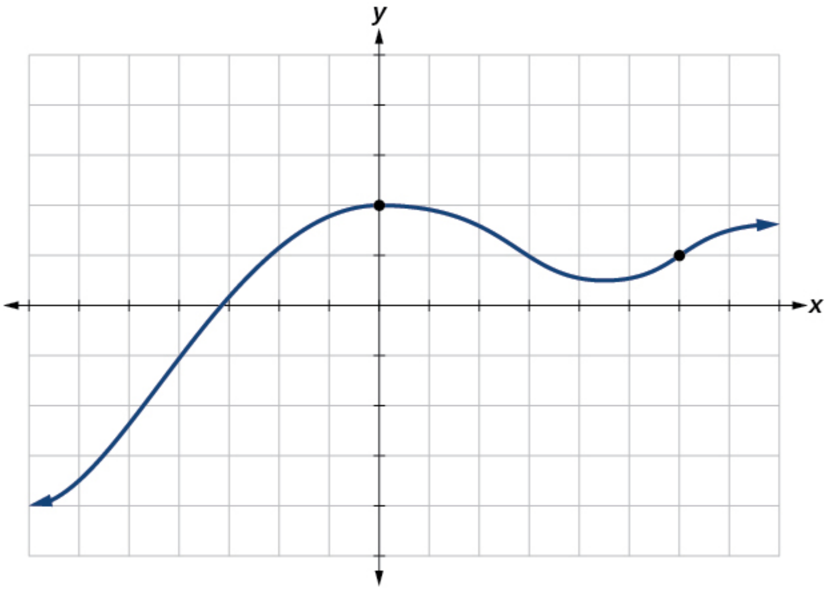
Solution:

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

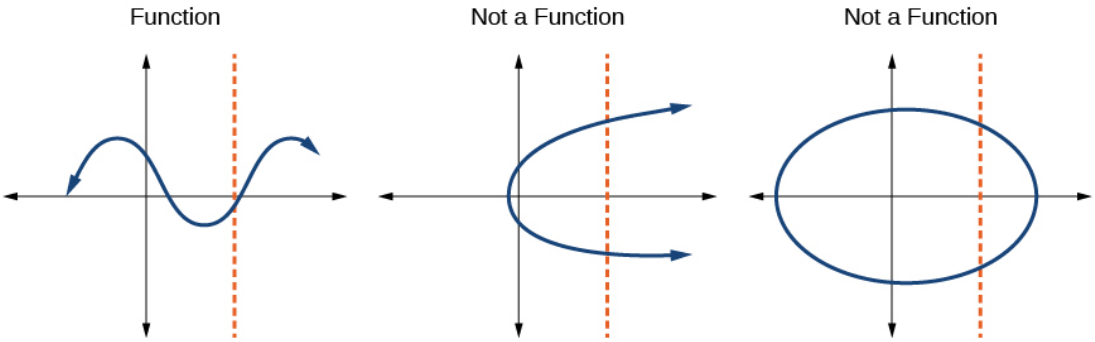
Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output value $y,$ and we say y is a function of $x,$ or $y=f\left(x\right)$ when the function is named $f.$ The graph of the function is the set of all points $\left(x,y\right)$ in the plane that satisfies the equation$y=f\left(x\right).$ If the function is defined for only a few input values, then the graph of the function consists of only a few points, where the *x*-coordinate of each point is an input value and the *y*-coordinate of each point is the corresponding output value. For example, the black dots on the graph in [link] tell us that $f\left(0\right)=2$ and $f\left(6\right)=1.$ However, the set of all points $\left(x,y\right)$ satisfying $y=f\left(x\right)$ is a curve. The curve shown includes $\left(0,2\right)$ and $\left(6,1\right)$ because the curve passes through those points.



The **vertical line test** can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does *not* define a function because a function has only one output value for each input value. See [link].



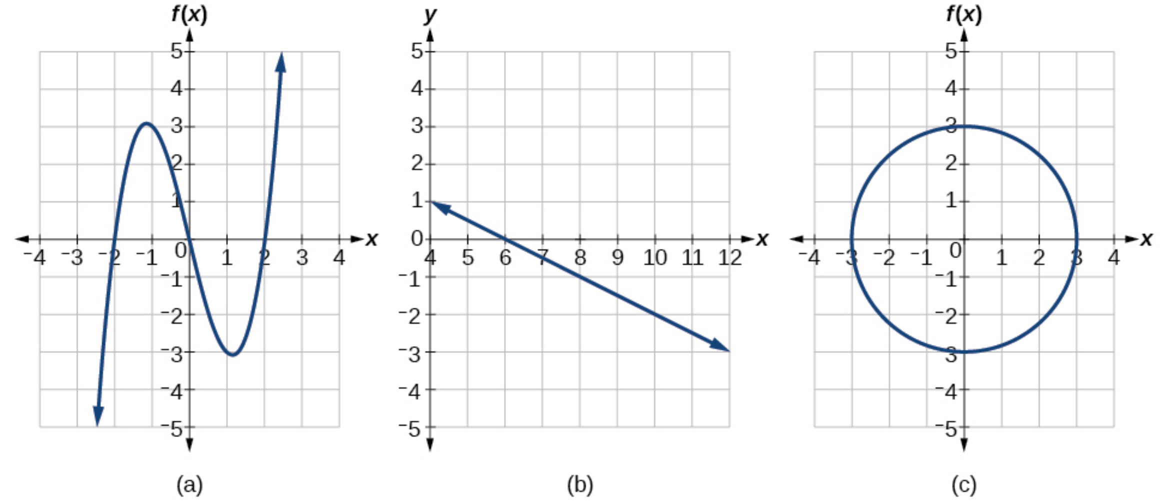
Note:
Given a graph, use the vertical line test to determine if the graph represents a function.

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

Example:
Exercise:

Problem:
Applying the Vertical Line Test

Which of the graphs in [link] represent(s) a function $y = f(x)$?



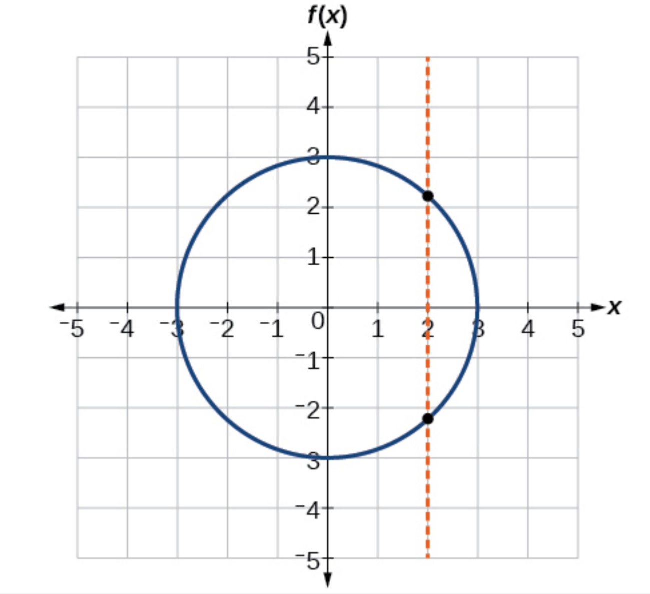
(a)

(b)

(c)

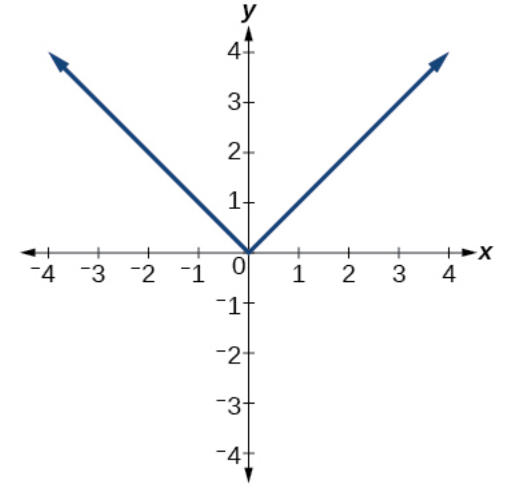
Solution:

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of [link]. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most *x*-values, a vertical line would intersect the graph at more than one point, as shown in [link].



Note:
Exercise:

Problem: Does the graph in [link] represent a function?



Solution:

yes

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Note:
Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function.

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

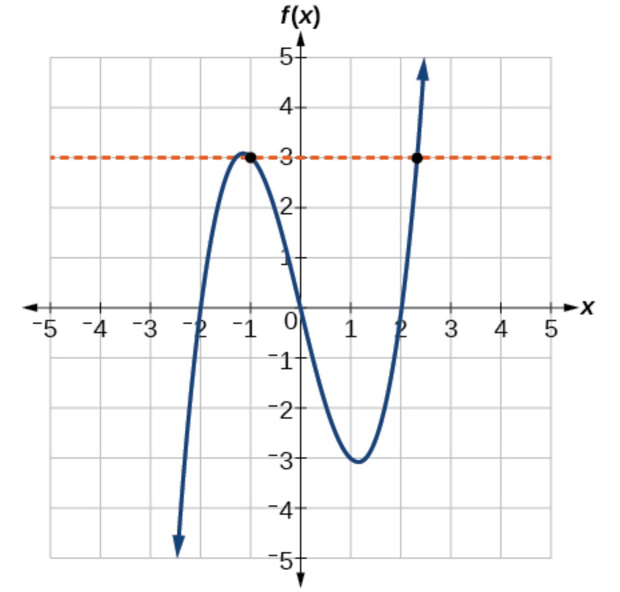
Example:
Exercise:

Problem:
Applying the Horizontal Line Test

Consider the functions shown in [link]**(a)** and [link]**(b)**. Are either of the functions one-to-one?

Solution:

The function in [link]**(a)** is not one-to-one. The horizontal line shown in [link] intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)



The function in [link]**(b)** is one-to-one. Any horizontal line will intersect a diagonal line at most once.

Note:
Exercise:

Problem: Is the graph shown in [link] one-to-one?

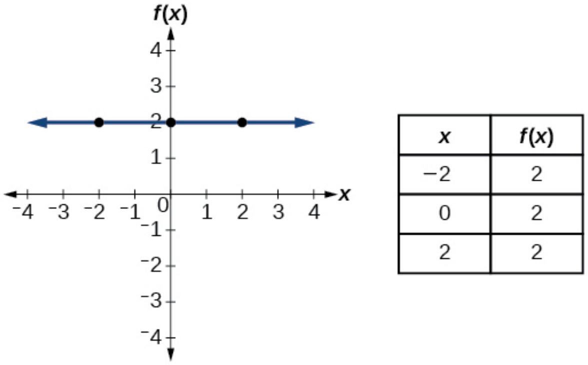
Solution:

No, because it does not pass the horizontal line test.

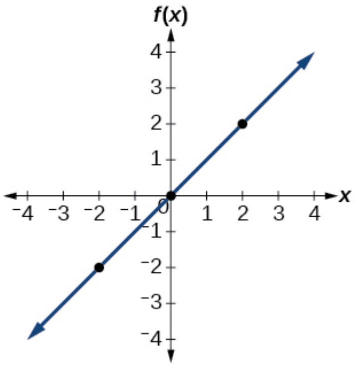
Identifying Basic Toolkit Functions

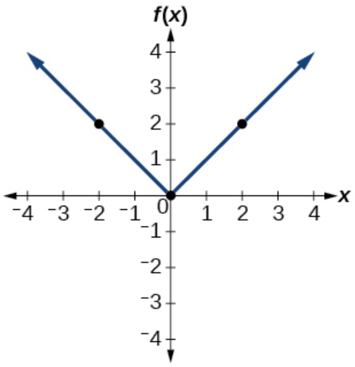
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our “toolkit functions,” which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f\left(x\right)$ as the output variable.

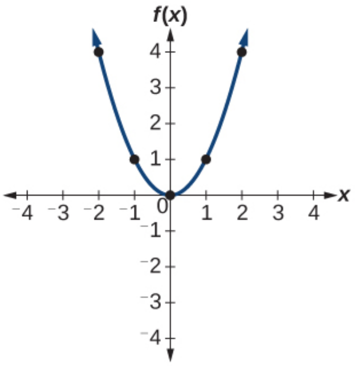
We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in [link].

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Constant</td><td>$f\left(x\right) = c$, where c is a constant</td><td>


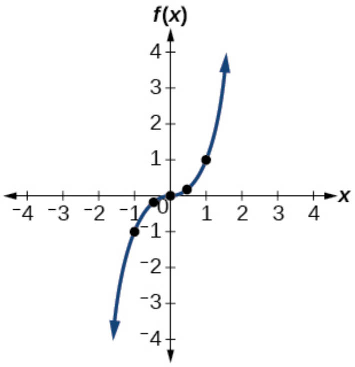
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>2</td></tr>
<tr><td>0</td><td>2</td></tr>
<tr><td>2</td><td>2</td></tr>
</table>
</td></tr>
</table>

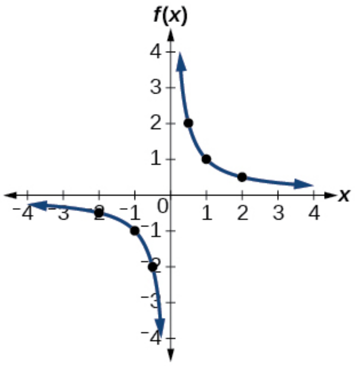
<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Identity</td><td>$f\left(x\right)=x$</td><td>

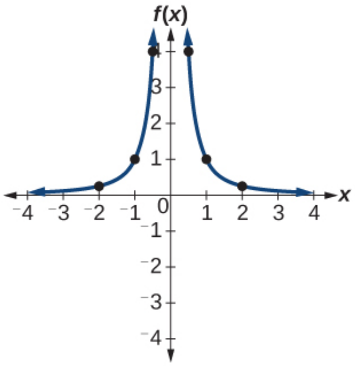
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−2</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>2</td><td>2</td></tr>
</table></td></tr>
<tr><td>Absolute value</td><td>$f\left(x\right)=\left|x\right|$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>2</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>2</td><td>2</td></tr>
</table></td></tr>
<tr><td>Quadratic</td><td>$f\left(x\right)={x}^{2}$</td><td>

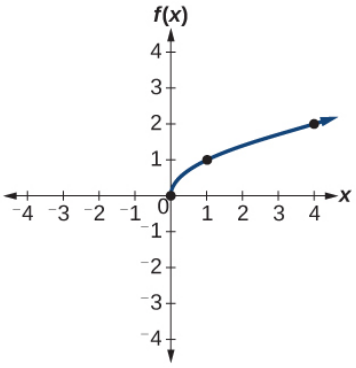
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>4</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>4</td></tr>
</table></td></tr>
</table>

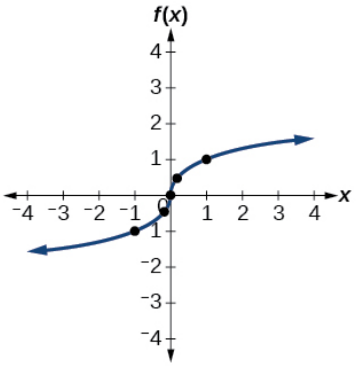
<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Cubic</td><td>$f\left(x\right)=x^3$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−0.125</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>0.5</td><td>0.125</td></tr>
<tr><td>1</td><td>1</td></tr>
</table></td></tr>
<tr><td>Reciprocal</td><td>$f\left(x\right)=\frac{1}{x}$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−0.5</td></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−2</td></tr>
<tr><td>0.5</td><td>2</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.5</td></tr>
</table></td></tr>
<tr><td>Reciprocal squared</td><td>$f\left(x\right)=\frac{1}{x^2}$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>0.25</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>−0.5</td><td>4</td></tr>
<tr><td>0.5</td><td>4</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.25</td></tr>
</table></td></tr>
</table>

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Square root</td><td>$f\left(x\right)=\sqrt{x}$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>4</td><td>2</td></tr>
</table></td></tr>
<tr><td>Cube root</td><td>$f\left(x\right)=\sqrt[3]{x}$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.125</td><td>−0.5</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>0.125</td><td>0.5</td></tr>
<tr><td>1</td><td>1</td></tr>
</table></td></tr>
</table>

Note:
Access the following online resources for additional instruction and practice with functions.

- Determine if a Relation is a Function
- Vertical Line Test
- Introduction to Functions
- Vertical Line Test on Graph
- One-to-one Functions
- Graphs as One-to-one Functions

Key Equations

Constant function	$f\left(x\right)=c$,where c is a constant
Identity function	$f\left(x\right)=x$
Absolute value function	$f\left(x\right)=\lvert x\rvert$
Quadratic function	$f\left(x\right)=x^2$
Cubic function	$f\left(x\right)=x^3$
Reciprocal function	$f\left(x\right)=\frac{1}{x}$
Reciprocal squared function	$f\left(x\right)=\frac{1}{x^2}$
Square root function	$f\left(x\right)=\sqrt{x}$
Cube root function	$f\left(x\right)=\sqrt[3]{x}$

Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output. See [link] and [link].
- Function notation is a shorthand method for relating the input to the output in the form $y=f\left(x\right).$ See [link] and [link].
- In tabular form, a function can be represented by rows or columns that relate to input and output values. See [link].
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value. See [link] and [link].
- To solve for a specific function value, we determine the input values that yield the specific output value. See [link].
- An algebraic form of a function can be written from an equation. See [link] and [link].
- Input and output values of a function can be identified from a table. See [link].
- Relating input values to output values on a graph is another way to evaluate a function. See [link].
- A function is one-to-one if each output value corresponds to only one input value. See [link].
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point. See [link].
- The graph of a one-to-one function passes the horizontal line test. See [link].

Section Exercises

Verbal

Exercise:

Problem: What is the difference between a relation and a function?

Solution:

A relation is a set of ordered pairs. A function is a special kind of relation in which no two ordered pairs have the same first coordinate.

Exercise:

Problem: What is the difference between the input and the output of a function?

Exercise:

Problem: Why does the vertical line test tell us whether the graph of a relation represents a function?

Solution:

When a vertical line intersects the graph of a relation more than once, that indicates that for that input there is more than one output. At any particular input value, there can be only one output if the relation is to be a function.

Exercise:

Problem: How can you determine if a relation is a one-to-one function?

Exercise:

Problem: Why does the horizontal line test tell us whether the graph of a function is one-to-one?

Solution:

When a horizontal line intersects the graph of a function more than once, that indicates that for that output there is more than one input. A function is one-to-one if each output corresponds to only one input.

Algebraic

For the following exercises, determine whether the relation represents a function.

Exercise:

Problem: $\{(a,b),\ \ (c,d),\ \ (a,c)\}$

Exercise:

Problem: $\{(a,b),(b,c),(c,c)\}$

Solution:

function

For the following exercises, determine whether the relation represents y as a function of x.

Exercise:

Problem: $5x+2y=10$

Exercise:

Problem: $y=x^2$

Solution:

function

Exercise:

Problem: $x = y^2$

Exercise:

Problem: $3x^2 + y = 14$

Solution:

function

Exercise:

Problem: $2x + y^2 = 6$

Exercise:

Problem: $y = -2x^2 + 40x$

Solution:

function

Exercise:

Problem: $y = \frac{1}{x}$

Exercise:

Problem: $x = \frac{3y+5}{7y-1}$

Solution:

function

Exercise:

Problem: $x = \sqrt{1 - y^2}$

Exercise:

Problem: $y = \frac{3x+5}{7x-1}$

Solution:

function

Exercise:

Problem: $x^2 + y^2 = 9$

Exercise:

Problem: $2xy = 1$

Solution:

function

Exercise:

Problem: $x = y^3$

Exercise:

Problem: $y = x^3$

Solution:

function

Exercise:

Problem: $y = \sqrt{1 - x^2}$

Exercise:

Problem: $x = \pm\sqrt{1 - y}$

Solution:

function

Exercise:

Problem: $y = \pm\sqrt{1 - x}$

Exercise:

Problem: $y^2 = x^2$

Solution:

not a function

Exercise:

Problem: $y^3 = x^2$

For the following exercises, evaluate the function f at the indicated values $f(-3), f(2), f(-a), -f(a), f(a + h)$.

Exercise:

Problem: $f(x) = 2x - 5$

Solution:

$f(-3) = -11; \quad f(2) = -1; \quad f(-a) = -2a - 5; \quad -f(a) = -2a + 5; \quad f(a + h) = 2a + 2h - 5$

Exercise:

Problem: $f(x) = -5x^2 + 2x - 1$

Exercise:

Problem: $f(x) = \sqrt{2-x} + 5$

Solution:

$f(-3) = \sqrt{5} + 5; \quad f(2) = 5; \quad f(-a) = \sqrt{2+a} + 5; \quad -f(a) = -\sqrt{2-a} - 5; \quad f(a+h) = \sqrt{2-a-h} + 5$

Exercise:

Problem: $f(x) = \frac{6x-1}{5x+2}$

Exercise:

Problem: $f(x) = |x-1| - |x+1|$

Solution:

$f(-3) = 2; \quad f(2) = 1 - 3 = -2; \quad f(-a) = |-a-1| - |-a+1|; \quad -f(a) = -|a-1| + |a+1|; \quad f(a$ -

Exercise:

Problem: Given the function $g(x) = 5 - x^2$, simplify $\frac{g(x+h)-g(x)}{h}$, $h \neq 0$.

Exercise:

Problem: Given the function $g(x) = x^2 + 2x$, simplify $\frac{g(x)-g(a)}{x-a}$, $x \neq a$.

Solution:

$\frac{g(x)-g(a)}{x-a} = x + a + 2, \; x \neq a$

Exercise:

Problem: Given the function $k(t) = 2t - 1$:

a. Evaluate $k(2)$.
b. Solve $k(t) = 7$.

Exercise:

Problem: Given the function $f(x) = 8 - 3x$:

a. Evaluate $f(-2)$.
b. Solve $f(x) = -1$.

Solution:

a. $f(-2) = 14$; b. $x = 3$

Exercise:

Problem: Given the function $p(c) = c^2 + c$:

a. Evaluate $p(-3)$.
b. Solve $p(c) = 2$.

Exercise:

Problem: Given the function $f(x) = x^2 - 3x$:

a. Evaluate $f(5)$.
b. Solve $f(x) = 4$.

Solution:

a. $f(5) = 10$; b. $x = -1$ or $x = 4$

Exercise:

Problem: Given the function $f(x) = \sqrt{x+2}$:

a. Evaluate $f(7)$.
b. Solve $f(x) = 4$.

Exercise:

Problem: Consider the relationship $3r + 2t = 18$.

a. Write the relationship as a function $r = f(t)$.
b. Evaluate $f(-3)$.
c. Solve $f(t) = 2$.

Solution:

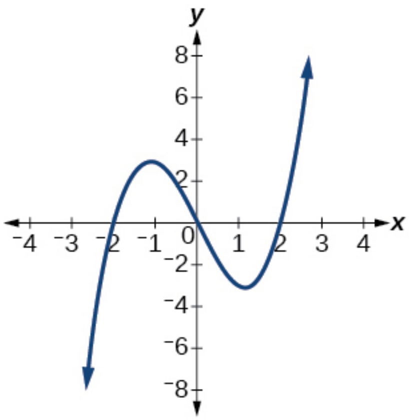
a. $f(t) = 6 - \frac{2}{3}t$; b. $f(-3) = 8$; c. $t = 6$

Graphical

For the following exercises, use the vertical line test to determine which graphs show relations that are functions.

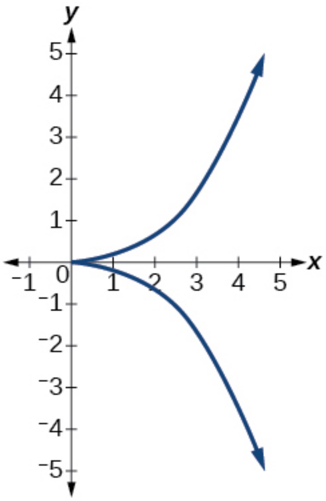
Exercise:

Problem:



Exercise:

Problem:

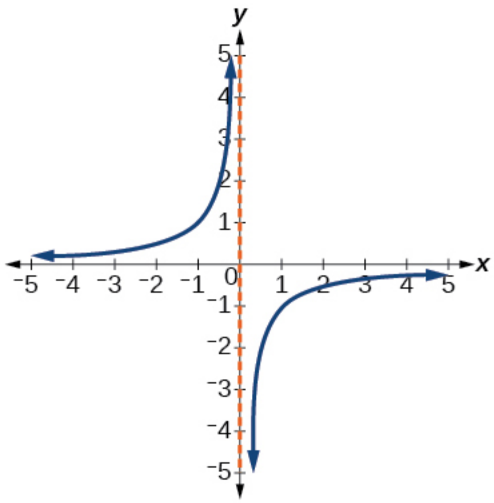


Solution:

not a function

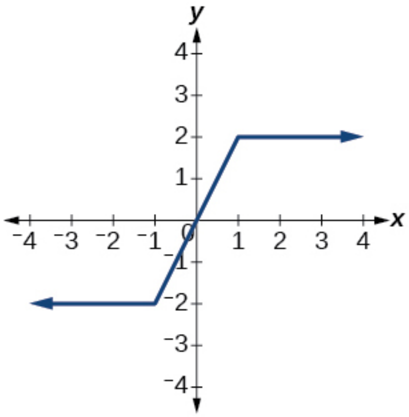
Exercise:

Problem:



Exercise:

Problem:

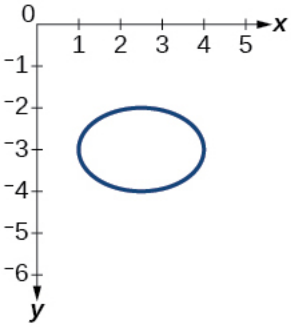


Solution:

function

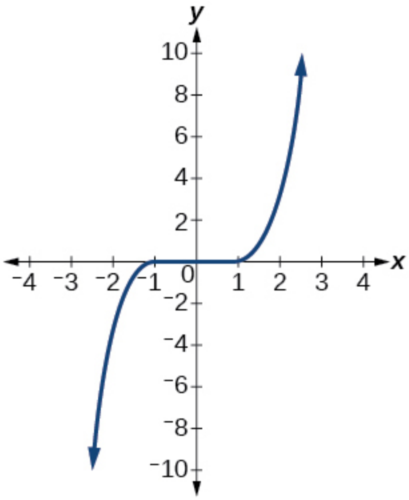
Exercise:

Problem:



Exercise:

Problem:

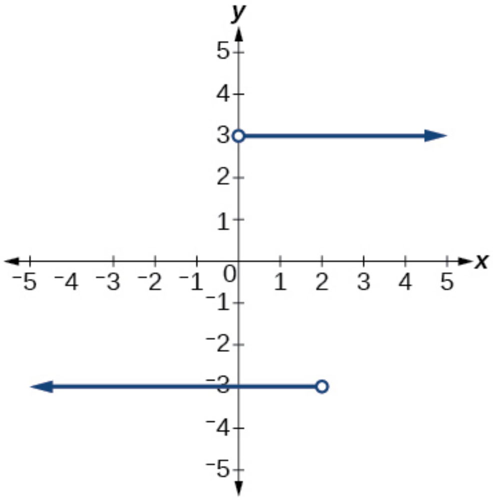


Solution:

function

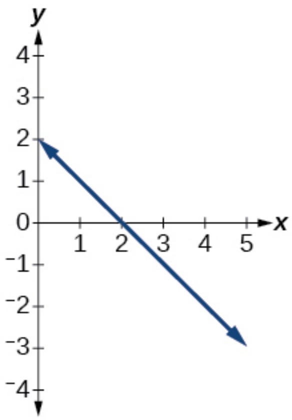
Exercise:

Problem:



Exercise:

Problem:

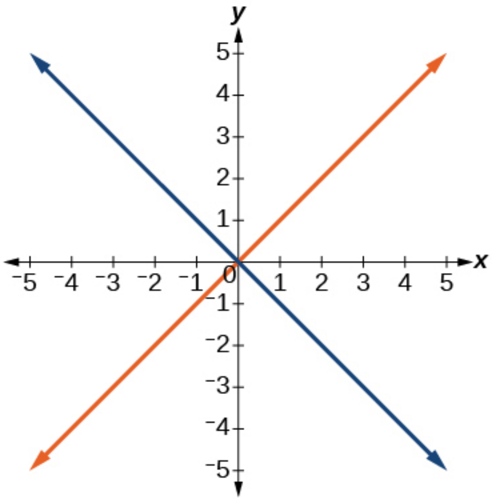


Solution:

function

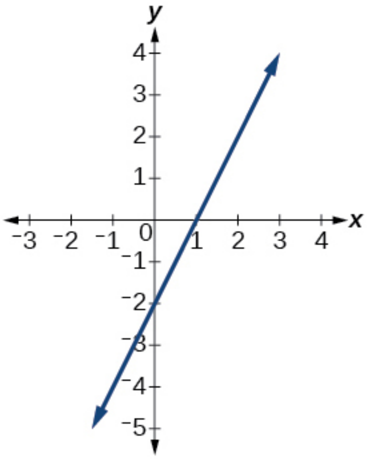
Exercise:

Problem:



Exercise:

Problem:

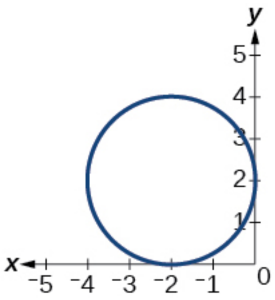


Solution:

function

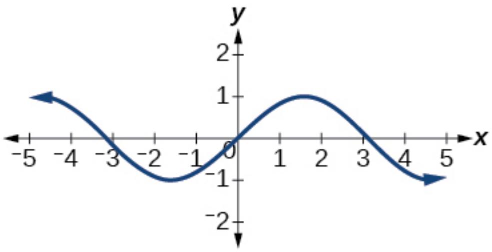
Exercise:

Problem:



Exercise:

Problem:



Solution:

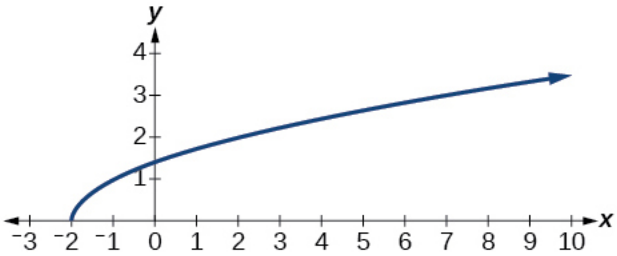
function

Exercise:

Problem: Given the following graph,

- Evaluate $f(-1)$.

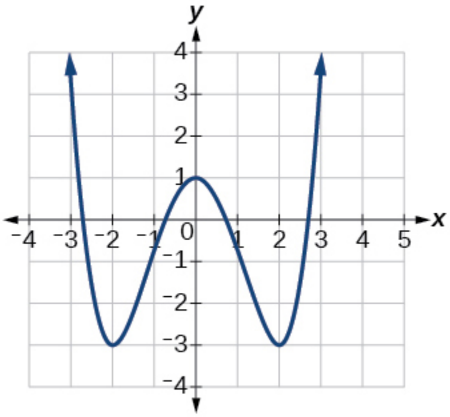
- Solve for $f(x) = 3$.



Exercise:

Problem: Given the following graph,

- Evaluate $f(0)$.
- Solve for $f(x) = -3$.



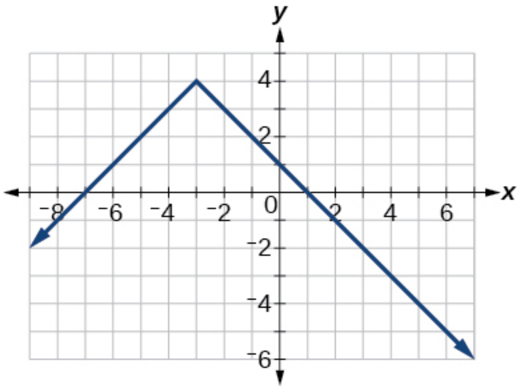
Solution:

a. $f(0) = 1$; b. $f(x) = -3$, $x = -2$ or $x = 2$

Exercise:

Problem: Given the following graph,

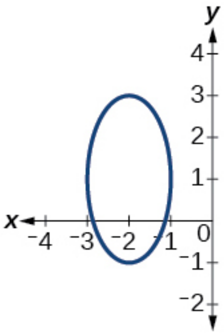
- Evaluate $f(4)$.
- Solve for $f(x) = 1$.



For the following exercises, determine if the given graph is a one-to-one function.

Exercise:

Problem:

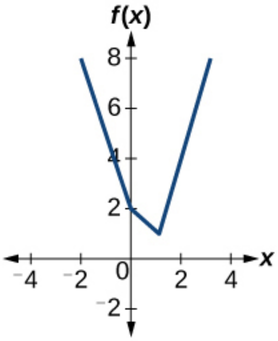


Solution:

not a function so it is also not a one-to-one function

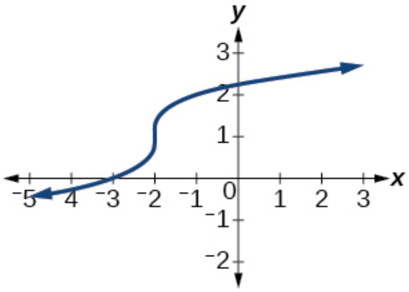
Exercise:

Problem:



Exercise:

Problem:

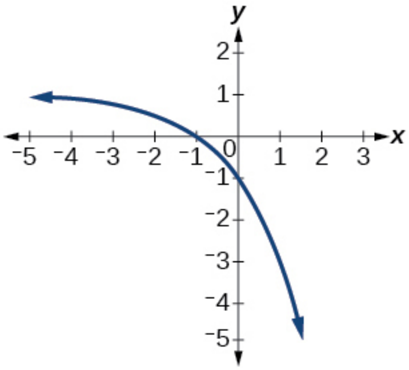


Solution:

one-to- one function

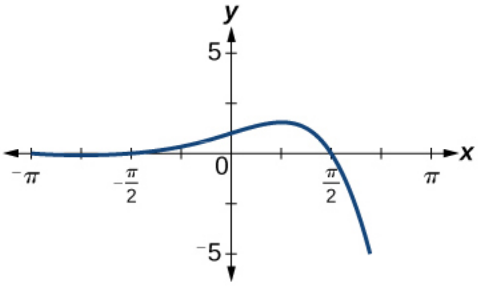
Exercise:

Problem:



Exercise:

Problem:



Solution:

function, but not one-to-one

Numeric

For the following exercises, determine whether the relation represents a function.

Exercise:

Problem: $\{(-1,-1),(-2,-2),(-3,-3)\}$

Exercise:

Problem: $\{(3,4),(4,5),(5,6)\}$

Solution:

function

Exercise:

Problem: $\{(2,5),(7,11),(15,8),(7,9)\}$

For the following exercises, determine if the relation represented in table form represents y as a function of x.

Exercise:

Problem:

x	5	10	15
y	3	8	14

Solution:

function

Exercise:

Problem:

x	5	10	15
y	3	8	8

Exercise:

Problem:

x	5	10	10

y	3	8	14

Solution:

not a function

For the following exercises, use the function f represented in [link].

x	$f(x)$
0	74
1	28
2	1
3	53
4	56
5	3
6	36
7	45
8	14
9	47

Exercise:

Problem: Evaluate $f(3)$.

Exercise:

Problem: Solve $f(x) = 1$.

Solution:

$f(x) = 1,\ x = 2$

For the following exercises, evaluate the function f at the values $f(-2),\ f(-1),\ f(0),\ f(1)$, and $f(2)$.

Exercise:

Problem: $f(x) = 4 - 2x$

Exercise:

Problem: $f\left(x\right)=8-3x$

Solution:

$f(-2)=14;\quad f(-1)=11;\quad f(0)=8;\quad f(1)=5;\quad f(2)=2$

Exercise:

Problem: $f\left(x\right)=8{x}^{2}-7x+3$

Exercise:

Problem: $f\left(x\right)=3+\sqrt{x+3}$

Solution:

$f(-2)=4;\qquad\qquad f(-1)=4.414;\quad f(0)=4.732;\quad f(1)=5;\quad f(2)=5.236$

Exercise:

Problem: $f(x)=\frac{x-2}{x+3}$

Exercise:

Problem: $f\left(x\right)={3}^{x}$

Solution:

$f(-2)=\frac{1}{9};\quad f(-1)=\frac{1}{3};\quad f(0)=1;\quad f(1)=3;\quad f(2)=9$

For the following exercises, evaluate the expressions, given functions $f,\;\, g,$and h:

- $f(x)=3x-2$
- $g(x)=5-{x}^{2}$
- $h(x)=-2{x}^{2}+3x-1$

Exercise:

Problem: $3f\left(1\right)-4g\left(-2\right)$

Exercise:

Problem:$f\left(\frac{7}{3}\right)-h\left(-2\right)$

Solution:

20

Technology

For the following exercises, graph $y={x}^{2}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise:

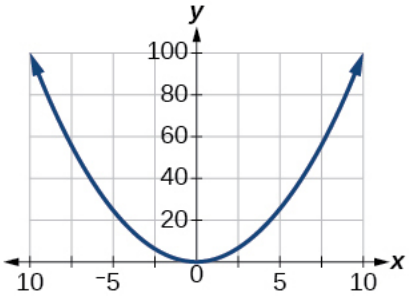
Problem: $[-0.1,\ 0.1]$

Exercise:

Problem:$[-10,\ 10]$

Solution:

$[0,\ 100]$



Exercise:

Problem: $[-100, 100]$

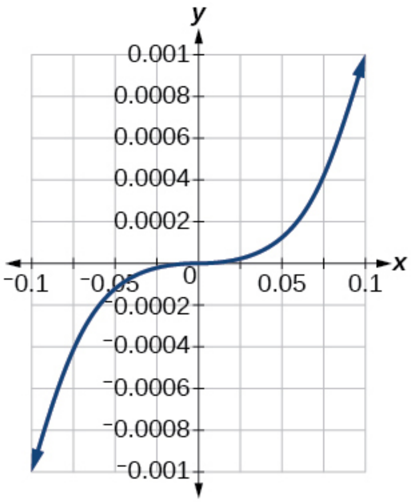
For the following exercises, graph $y = x^3$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise:

Problem: $[-0.1,\ 0.1]$

Solution:

$[-0.001,\ 0.001]$



Exercise:

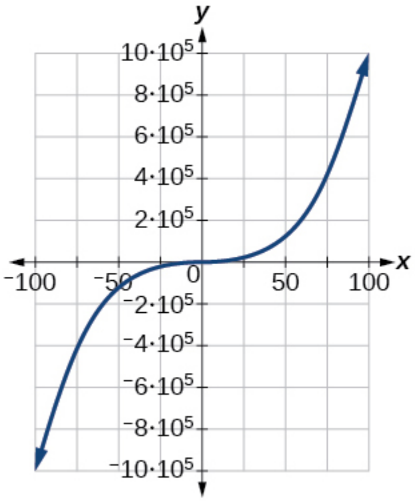
Problem: $[-10,\ 10]$

Exercise:

Problem:$[-100,\ 100]$

Solution:

$[-1,000,000,\ 1{,}000{,}000]$



For the following exercises, graph $y = \sqrt{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise:

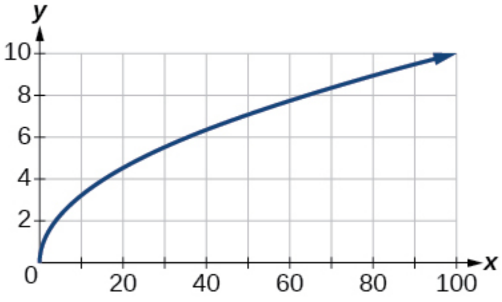
Problem: $[0,\ 0.01]$

Exercise:

Problem: $[0,\ 100]$

Solution:

$[0,\ 10]$



Exercise:

Problem: $[0,\ 10{,}000]$

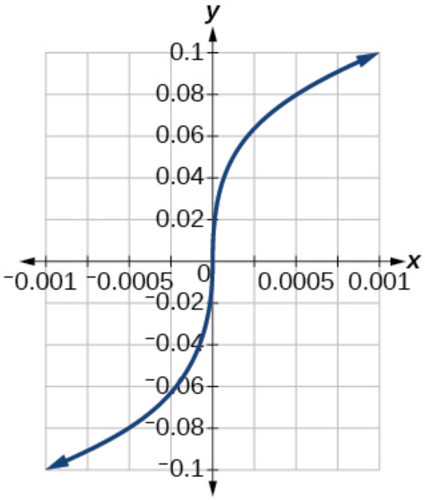
For the following exercises, graph $y=\sqrt[3]{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise:

Problem: $[-0.001, 0.001]$

Solution:

$[-0.1, 0.1]$



Exercise:

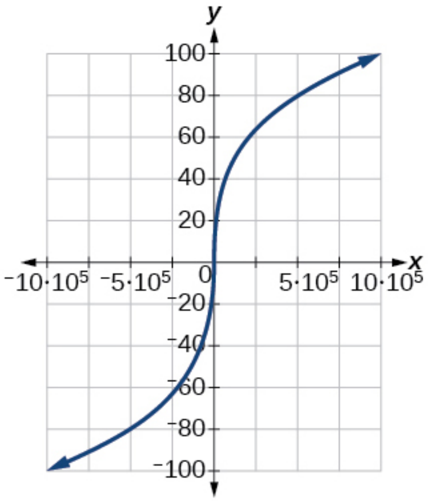
Problem: $[-1000, 1000]$

Exercise:

Problem: $[-1{,}000{,}000, 1{,}000{,}000]$

Solution:

$[-100,\ 100]$



Real-World Applications

Exercise:

Problem:

The amount of garbage, $G,$ produced by a city with population p is given by $G = f\left(p \right).$ G is measured in tons per week, and p is measured in thousands of people.

a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.
b. Explain the meaning of the statement $f\left(5 \right) = 2.$

Exercise:

Problem:

The number of cubic yards of dirt, $D,$ needed to cover a garden with area a square feet is given by $D = g\left(a \right).$

a. A garden with area 5000 ft^2 requires 50 yd^3 of dirt. Express this information in terms of the function g.
b. Explain the meaning of the statement $g\left(100 \right) = 1.$

Solution:

a. $g(5000) = 50$; b. The number of cubic yards of dirt required for a garden of 100 square feet is 1.

Exercise:

Problem:

Let $f\left(t \right)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:

a. $f\left(5 \right) = 30$
b. $f\left(10 \right) = 40$

Exercise:

Problem:

Let $h\left(t\right)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:

a. $h\left(1\right)=200$
b. $h\left(2\right)=350$

Solution:

a. The height of a rocket above ground after 1 second is 200 ft. b. the height of a rocket above ground after 2 seconds is 350 ft.

Exercise:

Problem: Show that the function $f\left(x\right)=3{\left(x-5\right)}^{2}+7$ is not one-to-one.

Glossary

dependent variable
an output variable

domain
the set of all possible input values for a relation

function
a relation in which each input value yields a unique output value

horizontal line test
a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

independent variable
an input variable

input
each object or value in a domain that relates to another object or value by a relationship known as a function

one-to-one function
a function for which each value of the output is associated with a unique input value

output
each object or value in the range that is produced when an input value is entered into a function

range
the set of output values that result from the input values in a relation

relation
a set of ordered pairs

vertical line test
a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

Linear Functions

In this section you will:

- Represent a linear function.
- Determine whether a linear function is increasing, decreasing, or constant.
- Interpret slope as a rate of change.
- Write and interpret an equation for a linear function.
- Graph linear functions.
- Determine whether lines are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.

Shanghai MagLev Train (credit: "kanegen"/Flickr)

Just as with the growth of a bamboo plant, there are many situations that involve constant change over time. Consider, for example, the first commercial maglev train in the world, the Shanghai MagLev Train ([link]). It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only eight minutes[footnote].
http://www.chinahighlights.com/shanghai/transportation/maglev-train.htm

Suppose a maglev train travels a long distance, and maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train's distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train's distance from the station at a given point in time.

Representing Linear Functions

The function describing the train’s motion is a linear function, which is defined as a function with a constant rate of change. This is a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train’s motion as a function using each method.

Representing a Linear Function in Word Form

Let’s begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.

- *The train’s distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.*

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the slope-intercept form of a line, where x is the input value, m is the rate of change, and b is the initial value of the dependent variable.

Equation:

$$\begin{array}{ll} \text{Equation form} & y = mx + b \\ \text{Function notation} & f(x) = mx + b \end{array}$$

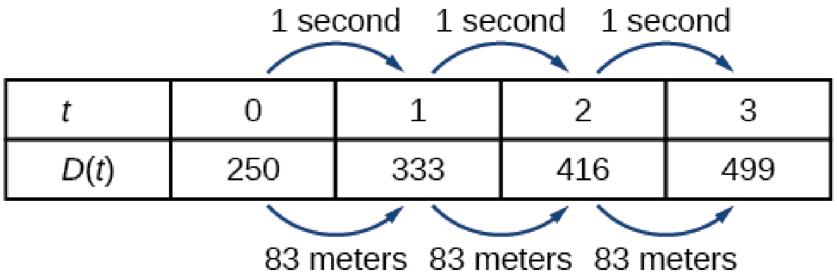
In the example of the train, we might use the notation $D(t)$ where the total distance D is a function of the time $t.$ The rate, $m,$ is 83 meters per second. The initial value of the dependent variable b is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.

Equation:

$$D(t) = 83t + 250$$

Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in [link]. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.



t	0	1	2	3
$D(t)$	250	333	416	499

1 second
1 second
1 second
83 meters
83 meters
83 meters

Tabular representation of the function D showing selected input and output values

Note:

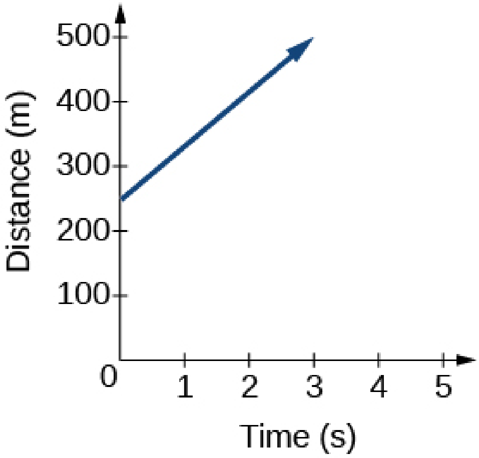
Can the input in the previous example be any real number?

No. The input represents time so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, $D(t) = 83t + 250,$ to draw a graph as represented in [link]. Notice the graph is a line**.** When we plot a linear function, the graph is always a line.

The rate of change, which is constant, determines the slant, or slope of the line. The point at which the input value is zero is the vertical intercept, or *y*-intercept, of the line. We can see from the graph that the *y*-intercept in the train example we just saw is $(0, 250)$ and represents the distance of the train from the station when it began moving at a constant speed.



The graph of $D(t) = 83t + 250$. Graphs of linear functions are lines because the rate of change is constant.

Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x) = 2x + 1.$ Ask yourself what numbers can be input to the function. In other words, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

Note:
Linear Function
A **linear function** is a function whose graph is a line. Linear functions can be written in the **slope-intercept form** of a line
Equation:

$$f(x) = mx + b$$

where b is the initial or starting value of the function (when input, $x = 0$), and m is the constant rate of change, or slope of the function. The *y*-intercept is at $(0, b)$.

Example:
Exercise:

Problem:
Using a Linear Function to Find the Pressure on a Diver

The pressure, P,in pounds per square inch (PSI) on the diver in [link] depends upon her depth below the water surface, d,in feet. This relationship may be modeled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.

(credit: Ilse Reijs and Jan-Noud Hutten)

Solution:

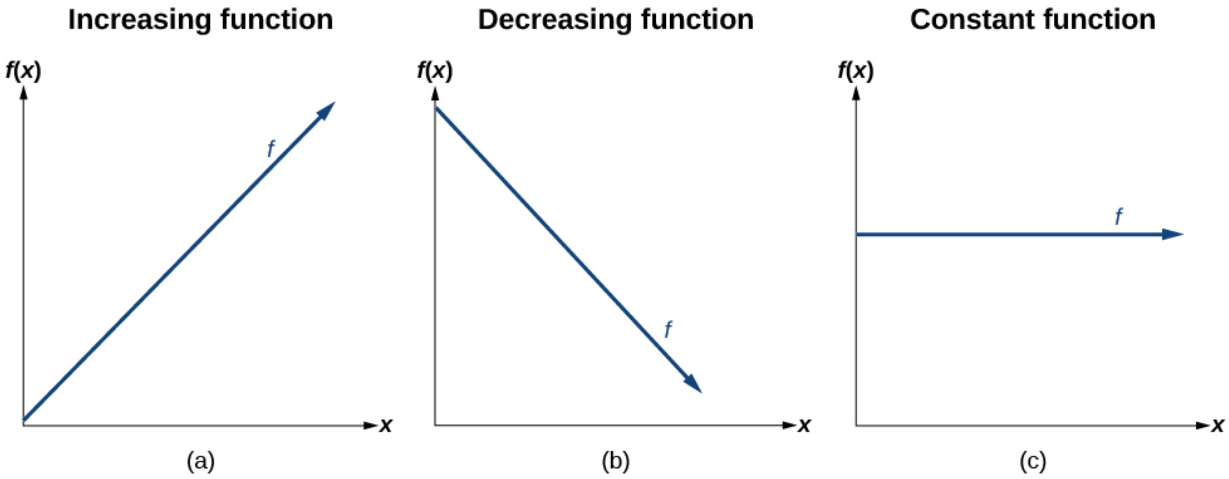
To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

Analysis

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

Determining Whether a Linear Function Is Increasing, Decreasing, or Constant

The linear functions we used in the two previous examples increased over time, but not every linear function does. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an increasing function has a positive slope. A line with a positive slope slants upward from left to right as in [link]**(a)**. For a decreasing function, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in [link]**(b)**. If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in [link]**(c)**.



(a) (b) (c)

Note:
Increasing and Decreasing Functions
The slope determines if the function is an **increasing linear function**, a **decreasing linear function**, or a constant function.

- $f(x) = mx + b$ is an increasing function if $m > 0$.
- $f(x) = mx + b$ is a decreasing function if $m < 0$.
- $f(x) = mx + b$ is a constant function if $m = 0$.

Example:
Exercise:

Problem:
Deciding Whether a Function Is Increasing, Decreasing, or Constant

Some recent studies suggest that a teenager sends an average of 60 texts per day[footnote]. For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then, determine whether the graph of the function is increasing, decreasing, or constant. http://www.cbsnews.com/8301-501465_162-57400228-501465/teens-are-sending-60-texts-a-day-study-says/

a. The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and output is the total number of texts sent.
b. A teen has a limit of 500 texts per month in his or her data plan. The input is the number of days, and output is the total number of texts remaining for the month.
c. A teen has an unlimited number of texts in his or her data plan for a cost of $50 per month. The input is the number of days, and output is the total cost of texting each month.

Solution:

Analyze each function.

a. The function can be represented as $f(x)=60x$ where x is the number of days. The slope, 60, is positive so the function is increasing. This makes sense because the total number of texts increases with each day.
b. The function can be represented as $f(x)=500-60x$ where $x$is the number of days. In this case, the slope is negative so the function is decreasing. This makes sense because the number of texts remaining decreases each day and this function represents the number of texts remaining in the data plan after x days.
c. The cost function can be represented as $f(x)=50$ because the number of days does not affect the total cost. The slope is 0 so the function is constant.

Interpreting Slope as a Rate of Change

In the examples we have seen so far, the slope was provided to us. However, we often need to calculate the slope given input and output values. Recall that given two values for the input, $x_1$and x_2,and two corresponding values for the output, y_1 and y_2 —which can be represented by a set of points, (x_1, y_1) and (x_2, y_2)—we can calculate the slope m.

Equation:

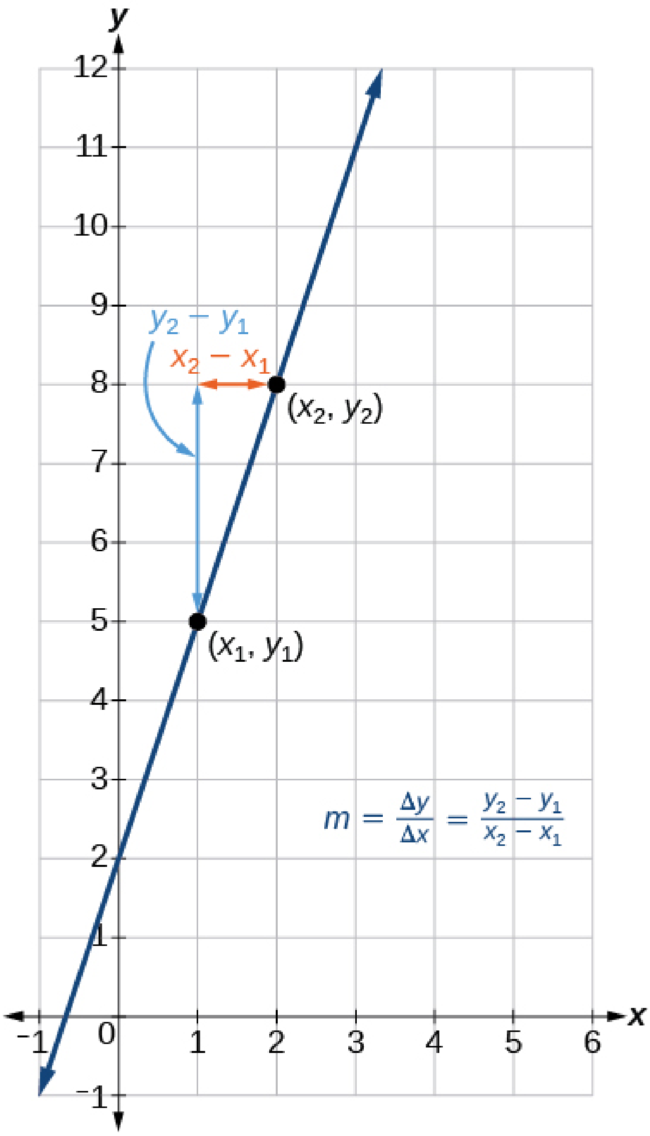
$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note that in function notation we can obtain two corresponding values for the output y_1 and y_2 for the function f, $y_1 = f(x_1)$ and $y_2 = f(x_2)$,so we could equivalently write

Equation:

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

[link] indicates how the slope of the line between the points, (x_1, y_1) and (x_2, y_2), is calculated. Recall that the slope measures steepness, or slant. The greater the absolute value of the slope, the steeper the slant is.



The slope of a function is calculated by the change in y divided by the change in $x.$ It does not matter which coordinate is used as the $(x_{2,}y_2)$ and which is the $(x_1, y_1),$ as long as each calculation is started with the elements from the same coordinate pair.

Note:

Are the units for slope always $\frac{\text{units for the output}}{\text{units for the input}}$?

Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

Note:

Calculate Slope

The slope, or rate of change, of a function m can be calculated according to the following:

Equation:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where x_1 and x_2 are input values, y_1 and y_2 are output values.

Note:

Given two points from a linear function, calculate and interpret the slope.

1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

Example:

Exercise:

Problem:

Finding the Slope of a Linear Function

If $f(x)$ is a linear function, and $(3, -2)$ and $(8, 1)$ are points on the line, find the slope. Is this function increasing or decreasing?

Solution:

The coordinate pairs are $(3, -2)$ and $(8, 1)$. To find the rate of change, we divide the change in output by the change in input.

Equation:

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{1-(-2)}{8-3} = \frac{3}{5}$$

We could also write the slope as $m = 0.6$. The function is increasing because $m > 0$.

Analysis

As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value, or *y*-coordinate, used corresponds with the first input value, or *x*-coordinate, used. Note that if we had reversed them, we would have obtained the same slope.

Equation:

$$m = \frac{(-2)-(1)}{3-8} = \frac{-3}{-5} = \frac{3}{5}$$

Note:

Exercise:

Problem:

If $f(x)$ is a linear function, and $(2, 3)$ and $(0, 4)$ are points on the line, find the slope. Is this function increasing or decreasing?

Solution:

$m = \frac{4-3}{0-2} = \frac{1}{-2} = -\frac{1}{2}$; decreasing because $m < 0$.

Example:

Exercise:

Problem:

Finding the Population Change from a Linear Function

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change of population per year if we assume the change was constant from 2008 to 2012.

Solution:

The rate of change relates the change in population to the change in time. The population increased by $27,800 - 23,400 = 4400$ people over the four-year time interval. To find the rate of change, divide the change in the number of people by the number of years.

Equation:

$$\frac{4{,}400 \text{ people}}{4 \text{ years}} = 1{,}100 \ \frac{\text{people}}{\text{year}}$$

So the population increased by 1,100 people per year.

Analysis

Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore reasonable.

Note:

Exercise:

Problem:

The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change of population per year if we assume the change was constant from 2009 to 2012.

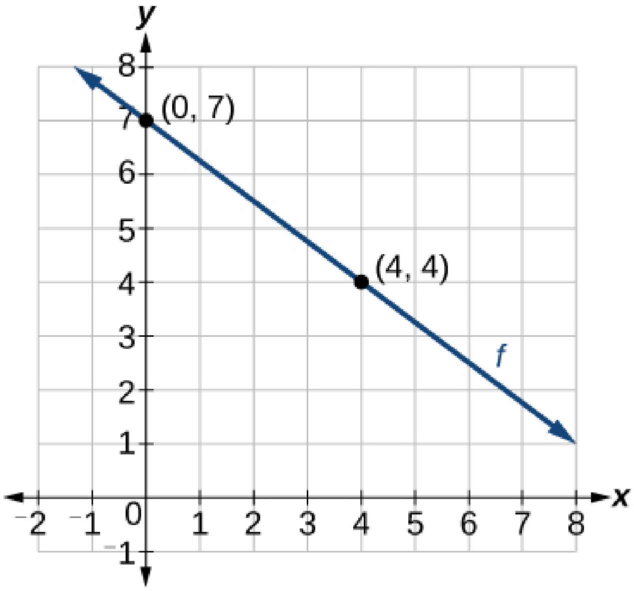
Solution:

Equation:

$$m = \frac{1,868 - 1,442}{2,012 - 2,009} = \frac{426}{3} = 142 \text{ people per year}$$

Writing and Interpreting an Equation for a Linear Function

Recall from Equations and Inequalities that we wrote equations in both the slope-intercept form and the point-slope form. Now we can choose which method to use to write equations for linear functions based on the information we are given. That information may be provided in the form of a graph, a point and a slope, two points, and so on. Look at the graph of the function f in [link].



We are not given the slope of the line, but we can choose any two points on the line to find the slope. Let’s choose $(0,7)$ and $(4,4)$.

Equation:

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{4-7}{4-0} \\ &= -\frac{3}{4} \end{aligned}$$

Now we can substitute the slope and the coordinates of one of the points into the point-slope form.

Equation:

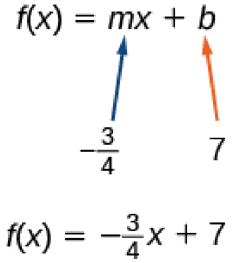
$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-4 &= -\frac{3}{4}(x-4) \end{aligned}$$

If we want to rewrite the equation in the slope-intercept form, we would find

Equation:

$$\begin{aligned} y-4 &= -\tfrac{3}{4}(x-4) \\ y-4 &= -\tfrac{3}{4}x+3 \\ y &= -\tfrac{3}{4}x+7 \end{aligned}$$

If we want to find the slope-intercept form without first writing the point-slope form, we could have recognized that the line crosses the *y*-axis when the output value is 7. Therefore, $b=7$. We now have the initial value b and the slope m so we can substitute m and b into the slope-intercept form of a line.



So the function is $f(x)=-\frac{3}{4}x+7$,and the linear equation would be $y=-\frac{3}{4}x+7$.

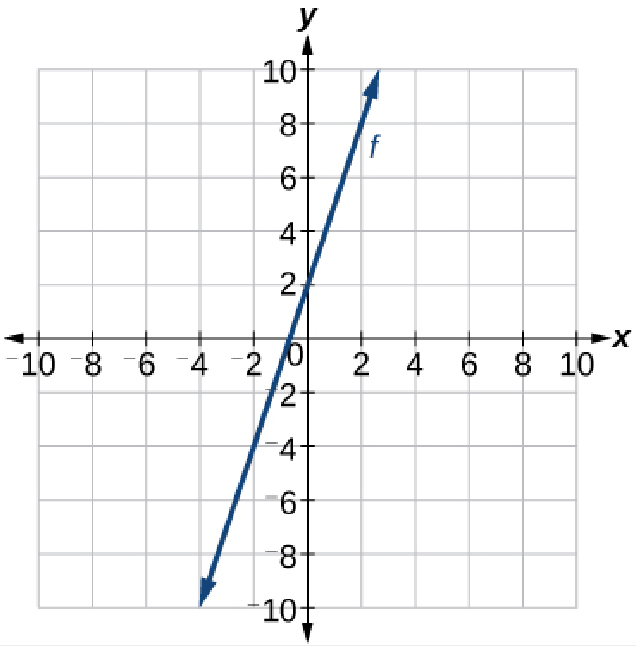
Note:
Given the graph of a linear function, write an equation to represent the function.

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the *y*-axis to identify the *y*-intercept by visual inspection.
4. Substitute the slope and *y*-intercept into the slope-intercept form of a line equation.

Example:
Exercise:

Problem:
Writing an Equation for a Linear Function

Write an equation for a linear function given a graph of f shown in [link].



Solution:

Identify two points on the line, such as $(0, 2)$ and $(-2, -4)$. Use the points to calculate the slope.

Equation:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-4 - 2}{-2 - 0} \\ &= \frac{-6}{-2} \\ &= 3 \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

Equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-4) &= 3(x - (-2)) \\ y + 4 &= 3(x + 2) \end{aligned}$$

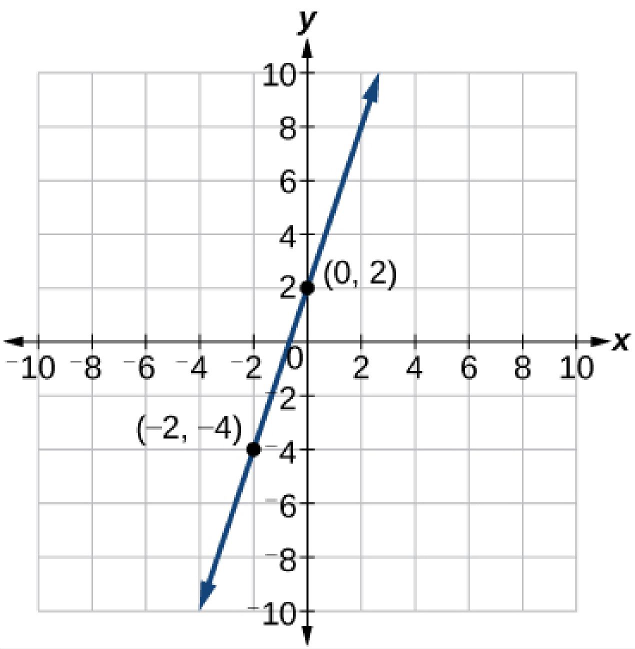
We can use algebra to rewrite the equation in the slope-intercept form.

Equation:

$$\begin{aligned} y+4 &= 3(x+2) \\ y+4 &= 3x+6 \\ y &= 3x+2 \end{aligned}$$

Analysis

This makes sense because we can see from [link] that the line crosses the *y*-axis at the point $(0,\ 2)$, which is the *y*-intercept, so $b=2$.



Example:
Exercise:

Problem:
Writing an Equation for a Linear Cost Function

Suppose Ben starts a company in which he incurs a fixed cost of $1,250 per month for the overhead, which includes his office rent. His production costs are $37.50 per item. Write a linear function C where $C\ (x)$ is the cost for x items produced in a given month.

Solution:

The fixed cost is present every month, $1,250. The costs that can vary include the cost to produce each item, which is $37.50. The variable cost, called the marginal cost, is represented by 37.5. The cost Ben incurs is the sum of these two costs, represented by $C\left(x\right)=1250+37.5x$.

Analysis

If Ben produces 100 items in a month, his monthly cost is found by substituting 100 for x.

Equation:

$$\begin{array}{rcl}C(100) & = & 1250+37.5(100)\\ & = & 5000\end{array}$$

So his monthly cost would be \$5,000.

Example:

Exercise:

Problem:

Writing an Equation for a Linear Function Given Two Points

If f is a linear function, with $f(3)=-2$,and $f(8)=1$,find an equation for the function in slope-intercept form.

Solution:

We can write the given points using coordinates.

Equation:

$$\begin{array}{rcl}f(3) & = & -2\to(3,-2)\\ f(8) & = & 1\to(8,1)\end{array}$$

We can then use the points to calculate the slope.

Equation:

$$\begin{array}{rcl}m & = & \frac{y_2-y_1}{x_2-x_1}\\ & = & \frac{1-(-2)}{8-3}\\ & = & \frac{3}{5}\end{array}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

Equation:

$$\begin{array}{rcl} y-y_1 & = & m\left(x-x_1\right) \\ y-(-2) & = & \frac{3}{5}(x-3) \end{array}$$

We can use algebra to rewrite the equation in the slope-intercept form.

Equation:

$$\begin{array}{rcl} y+2 & = & \frac{3}{5}(x-3) \\ y+2 & = & \frac{3}{5}x-\frac{9}{5} \\ y & = & \frac{3}{5}x-\frac{19}{5} \end{array}$$

Note:

Exercise:

Problem:

If $f(x)$ is a linear function, with $f(2)=–11$,and $f(4)=-25$,write an equation for the function in slope-intercept form.

Solution:

$y=-7x+3$

Modeling Real-World Problems with Linear Functions

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know, or can figure out, the initial value and the rate of change of a linear function, we can solve many different kinds of real-world problems.

Note:

Given a linear function f and the initial value and rate of change, evaluate $f\left(c\right)$.

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into $f(x)=mx+b$.
3. Evaluate the function at $x=c$.

Example:
Exercise:

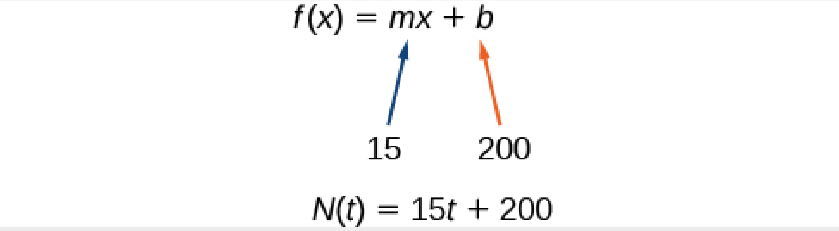
Problem:
Using a Linear Function to Determine the Number of Songs in a Music Collection

Marcus currently has 200 songs in his music collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N,in his collection as a function of time, t,the number of months. How many songs will he own at the end of one year?

Solution:

The initial value for this function is 200 because he currently owns 200 songs, so $N(0)=200$,which means that $b=200$.

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. Therefore we know that $m=15$. We can substitute the initial value and the rate of change into the slope-intercept form of a line.



We can write the formula $N(t)=15t+200$.

With this formula, we can then predict how many songs Marcus will have at the end of one year (12 months). In other words, we can evaluate the function at $t=12$.

Equation:

$$\begin{aligned} N(12) &= 15(12)+200 \\ &= 180+200 \\ &= 380 \end{aligned}$$

Marcus will have 380 songs in 12 months.

Analysis

Notice that N is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

Example:
Exercise:

Problem:
Using a Linear Function to Calculate Salary Based on Commission

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya's weekly income I,depends on the number of new policies, n,he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies and earned \$920. Find an equation for $I(n)$,and interpret the meaning of the components of the equation.

Solution:

The given information gives us two input-output pairs: $(3,760)$ and $(5,920).$ We start by finding the rate of change.

Equation:

$$\begin{aligned} m &= \frac{920-760}{5-3} \\ &= \frac{\$160}{2\text{ policies}} \\ &= \$80\text{ per policy} \end{aligned}$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy. Therefore, Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value.

Equation:

$$\begin{array}{rcl} I(n) & = & 80n + b \\ 760 & = & 80(3) + b \qquad \text{when}\, n = 3, I(3) = 760 \\ 760 & - & 80(3) = b \\ 520 & = & b \end{array}$$

The value of b is the starting value for the function and represents Ilya’s income when $n = 0,$ or when no new policies are sold. We can interpret this as Ilya’s base salary for the week, which does not depend upon the number of policies sold.

We can now write the final equation.

Equation:

$$I(n) = 80n + 520$$

Our final interpretation is that Ilya’s base salary is $520 per week and he earns an additional $80 commission for each policy sold.

Example:

Exercise:

Problem:

Using Tabular Form to Write an Equation for a Linear Function

[link] relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.

number of weeks, *w*	0	2	4	6
number of rats, *P(w)*	1000	1080	1160	1240

Solution:

We can see from the table that the initial value for the number of rats is 1000, so $b = 1000$.

Rather than solving for m,we can tell from looking at the table that the population increases by 80 for every 2 weeks that pass. This means that the rate of change is 80 rats per 2 weeks, which can be simplified to 40 rats per week.

Equation:

$$P(w) = 40w + 1000$$

If we did not notice the rate of change from the table we could still solve for the slope using any two points from the table. For example, using $(2, 1080)$ and $(6, 1240)$

Equation:

$$\begin{aligned} m &= \tfrac{1240-1080}{6-2} \\ &= \tfrac{160}{4} \\ &= 40 \end{aligned}$$

Note:

Is the initial value always provided in a table of values like [link]?

No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to 0, find the slope, substitute one coordinate pair and the slope into $f(x) = mx + b$, and solve for b.

Note:

Exercise:

Problem:

A new plant food was introduced to a young tree to test its effect on the height of the tree. [link] shows the height of the tree, in feet, x months since the measurements began. Write a linear function, $H(x)$,where x is the number of months since the start of the experiment.

x	0	2	4	8	12
***H*(*x*)**	12.5	13.5	14.5	16.5	18.5

Solution:

$H\left(x\right)=0.5x+12.5$

Graphing Linear Functions

Now that we’ve seen and interpreted graphs of linear functions, let’s take a look at how to create the graphs. There are three basic methods of graphing linear functions. The first is by plotting points and then drawing a line through the points. The second is by using the *y*-intercept and slope. And the third method is by using transformations of the identity function $f(x)=x$.

Graphing a Function by Plotting Points

To find points of a function, we can choose input values, evaluate the function at these input values, and calculate output values. The input values and corresponding output values form coordinate pairs. We then plot the coordinate pairs on a grid. In general, we should evaluate the function at a minimum of two inputs in order to find at least two points on the graph. For example, given the function, $f(x)=2x$,we might use the input values 1 and 2. Evaluating the function for an input value of 1 yields an output value of 2, which is represented by the point $(1,2)$. Evaluating the function for an input value of 2 yields an output value of 4, which is represented by the point $(2,4)$. Choosing three points is often advisable because if all three points do not fall on the same line, we know we made an error.

Note:
Given a linear function, graph by plotting points.

1. Choose a minimum of two input values.
2. Evaluate the function at each input value.
3. Use the resulting output values to identify coordinate pairs.
4. Plot the coordinate pairs on a grid.

5. Draw a line through the points.

Example:
Exercise:

Problem:
Graphing by Plotting Points

Graph $f(x)=-\frac{2}{3}x+5$ by plotting points.

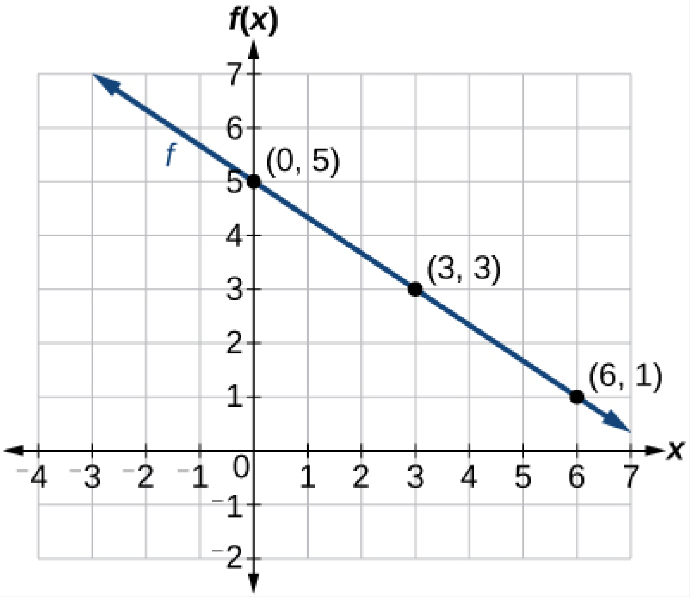
Solution:

Begin by choosing input values. This function includes a fraction with a denominator of 3, so let’s choose multiples of 3 as input values. We will choose 0, 3, and 6.

Evaluate the function at each input value, and use the output value to identify coordinate pairs.
Equation:

$$\begin{array}{ll} x=0 & f(0)=-\frac{2}{3}(0)+5=5\Rightarrow(0,5) \\ x=3 & f(3)=-\frac{2}{3}(3)+5=3\Rightarrow(3,3) \\ x=6 & f(6)=-\frac{2}{3}(6)+5=1\Rightarrow(6,1) \end{array}$$

Plot the coordinate pairs and draw a line through the points. [link] represents the graph of the function $f(x)=-\frac{2}{3}x+5$.



The graph of the linear function $f(x) = -\frac{2}{3}x + 5$.

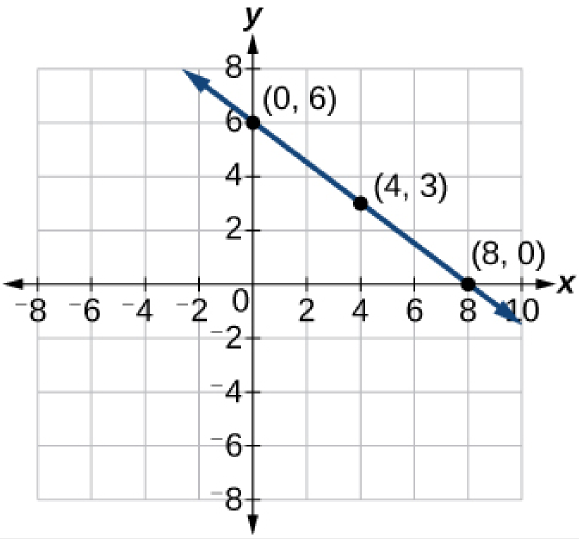
Analysis

The graph of the function is a line as expected for a linear function. In addition, the graph has a downward slant, which indicates a negative slope. This is also expected from the negative, constant rate of change in the equation for the function.

Note:
Exercise:

Problem: Graph $f(x) = -\frac{3}{4}x + 6$ by plotting points.

Solution:



Graphing a Function Using *y*-intercept and Slope

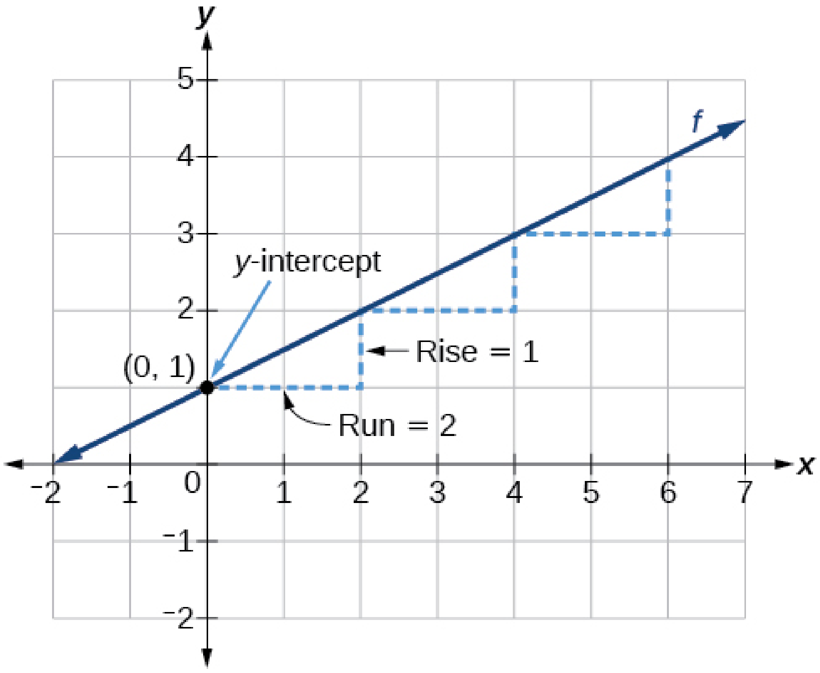
Another way to graph linear functions is by using specific characteristics of the function rather than plotting points. The first characteristic is its *y*-intercept, which is the point at which the input value is zero. To find the *y*-intercept, we can set $x=0$ in the equation.

The other characteristic of the linear function is its slope**.**

Let’s consider the following function.
Equation:

$$f(x)=\frac{1}{2}x+1$$

The slope is $\frac{1}{2}$. Because the slope is positive, we know the graph will slant upward from left to right. The *y*-intercept is the point on the graph when $x=0$. The graph crosses the *y*-axis at $(0,1)$. Now we know the slope and the *y*-intercept. We can begin graphing by plotting the point $(0,1)$. We know that the slope is the change in the *y*-coordinate over the change in the *x*-coordinate. This is commonly referred to as rise over run, $m=\frac{\text{rise}}{\text{run}}$. From our example, we have $m=\frac{1}{2}$,which means that the rise is 1 and the run is 2. So starting from our *y*-intercept $(0,1)$,we can rise 1 and then run 2, or run 2 and then rise 1. We repeat until we have a few points, and then we draw a line through the points as shown in [link].



Note:

Graphical Interpretation of a Linear Function

In the equation $f(x) = mx + b$

- b is the *y*-intercept of the graph and indicates the point $(0, b)$ at which the graph crosses the *y*-axis.
- m is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:

Equation:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note:

Do all linear functions have *y*-intercepts?

Yes. All linear functions cross the y-axis and therefore have y-intercepts. (Note: *A vertical line is parallel to the y-axis does not have a y-intercept, but it is not a function*.)

Note:

Given the equation for a linear function, graph the function using the *y*-intercept and slope.

1. Evaluate the function at an input value of zero to find the *y*-intercept.
2. Identify the slope as the rate of change of the input value.
3. Plot the point represented by the *y*-intercept.
4. Use $\frac{\text{rise}}{\text{run}}$ to determine at least two more points on the line.
5. Sketch the line that passes through the points.

Example:

Exercise:

Problem:

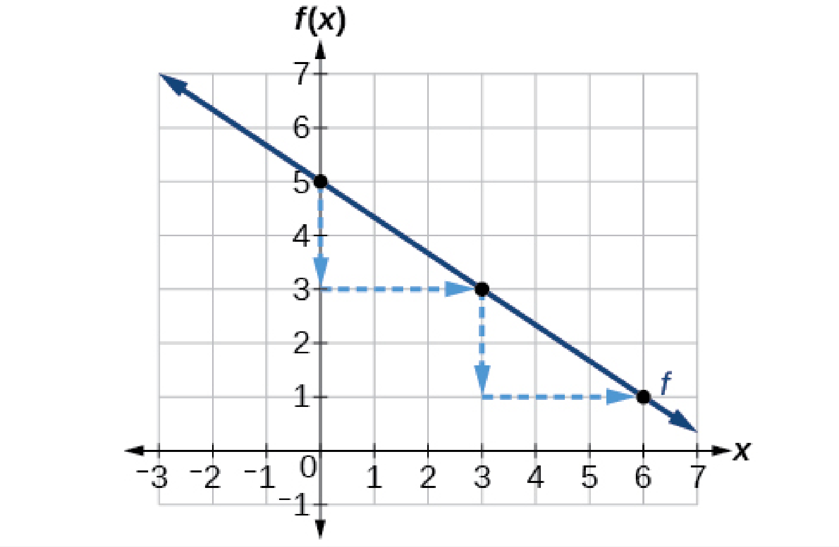
Graphing by Using the *y*-intercept and Slope

Graph $f(x)=-\frac{2}{3}x+5$ using the *y*-intercept and slope.

Solution:

Evaluate the function at $x=0$ to find the *y*-intercept. The output value when $x=0$ is 5, so the graph will cross the *y*-axis at $(0,5)$.

According to the equation for the function, the slope of the line is $-\frac{2}{3}$. This tells us that for each vertical decrease in the “rise” of -2 units, the “run” increases by 3 units in the horizontal direction. We can now graph the function by first plotting the *y*-intercept on the graph in [link]. From the initial value $(0,5)$ we move down 2 units and to the right 3 units. We can extend the line to the left and right by repeating, and then drawing a line through the points.



Graph of $f(x) = -2/3x + 5$ and shows how to calculate the rise over run for the slope.

Analysis

The graph slants downward from left to right, which means it has a negative slope as expected.

Note:
Exercise:

Problem: Find a point on the graph we drew in [link] that has a negative *x*-value.

Solution:

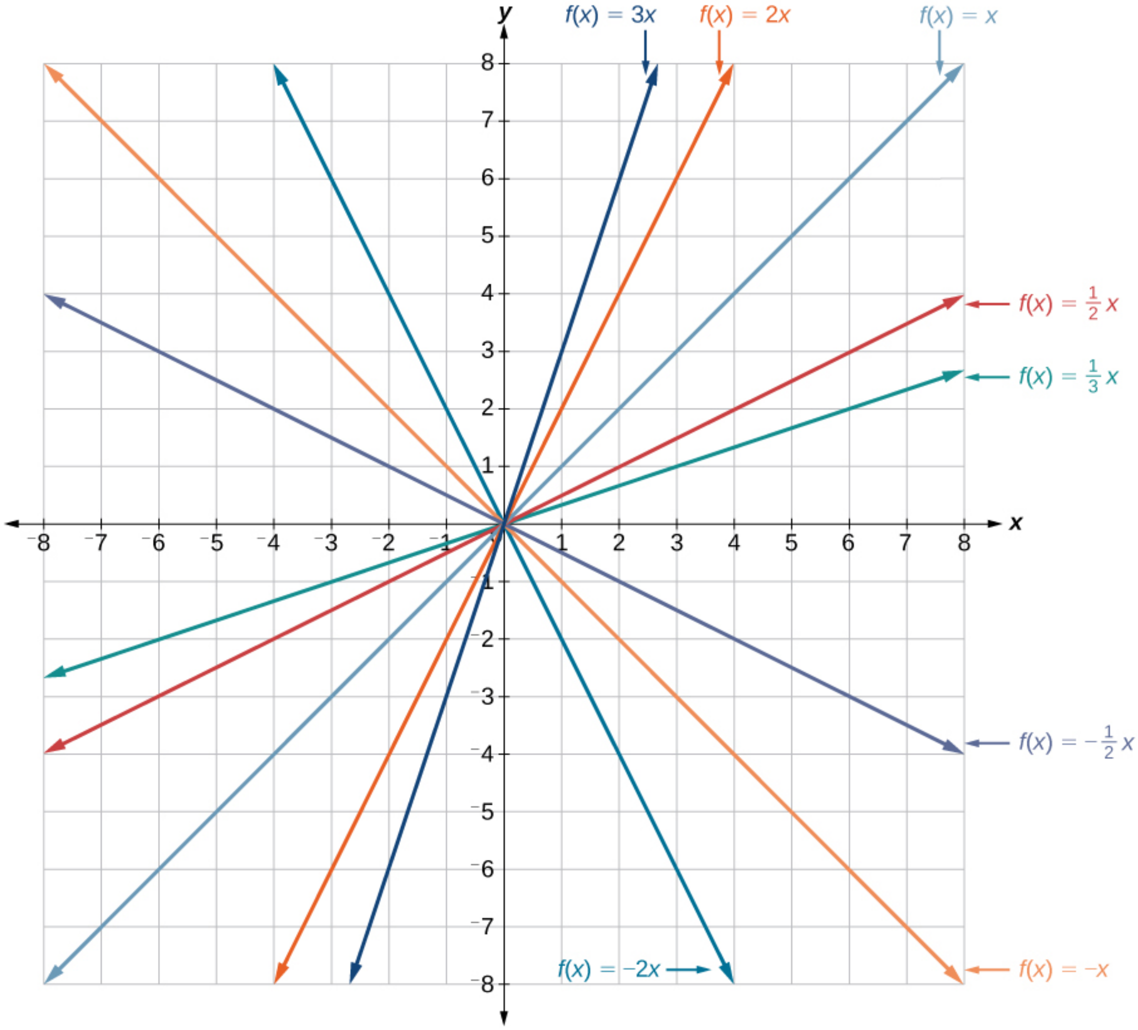
Possible answers include $(-3, 7)$, $(-6, 9)$, or $(-9, 11)$.

Graphing a Function Using Transformations

Another option for graphing is to use a transformation of the identity function $f(x) = x$. A function may be transformed by a shift up, down, left, or right. A function may also be transformed using a reflection, stretch, or compression.

Vertical Stretch or Compression

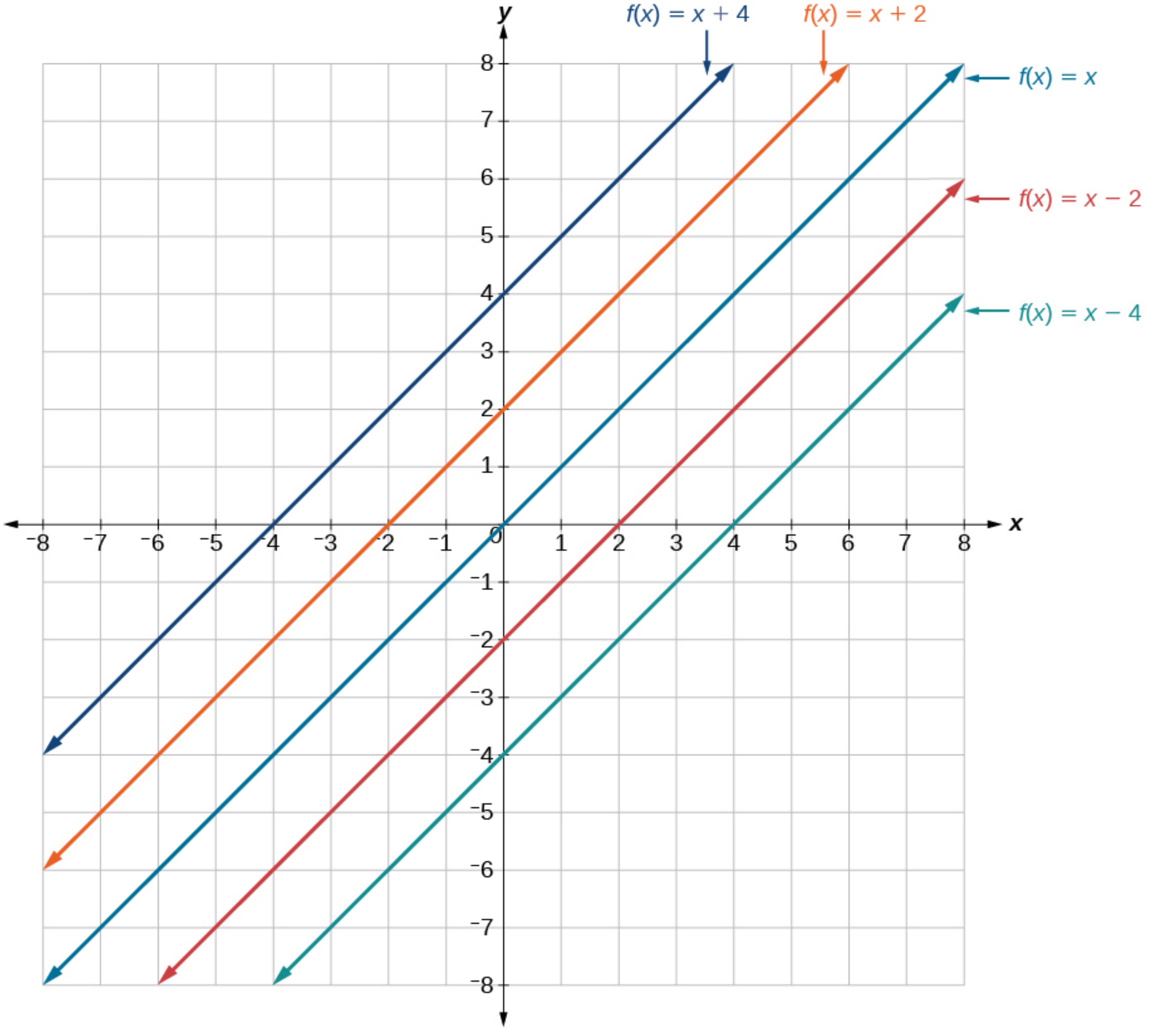
In the equation $f(x) = mx$,the m is acting as the vertical stretch or compression of the identity function. When m is negative, there is also a vertical reflection of the graph. Notice in [link] that multiplying the equation of $f(x) = x$ by m stretches the graph of f by a factor of m units if $m > 1$ and compresses the graph of f by a factor of m units if $0 < m < 1$. This means the larger the absolute value of $m,$ the steeper the slope.



Vertical stretches and compressions and reflections on the function $f(x) = x$

Vertical Shift

In $f(x) = mx + b$,the b acts as the vertical shift, moving the graph up and down without affecting the slope of the line. Notice in [link] that adding a value of b to the equation of $f(x) = x$ shifts the graph of f a total of b units up if b is positive and $|b|$ units down if b is negative.



This graph illustrates vertical shifts of the function $f(x) = x$.

Using vertical stretches or compressions along with vertical shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way to graph this type of function, it is still important to practice each method.

Note:

Given the equation of a linear function, use transformations to graph the linear function in the form $f\left(x\right)=mx+b$.

1. Graph $f\left(x\right)=x$.
2. Vertically stretch or compress the graph by a factor m.
3. Shift the graph up or down b units.

Example:

Exercise:

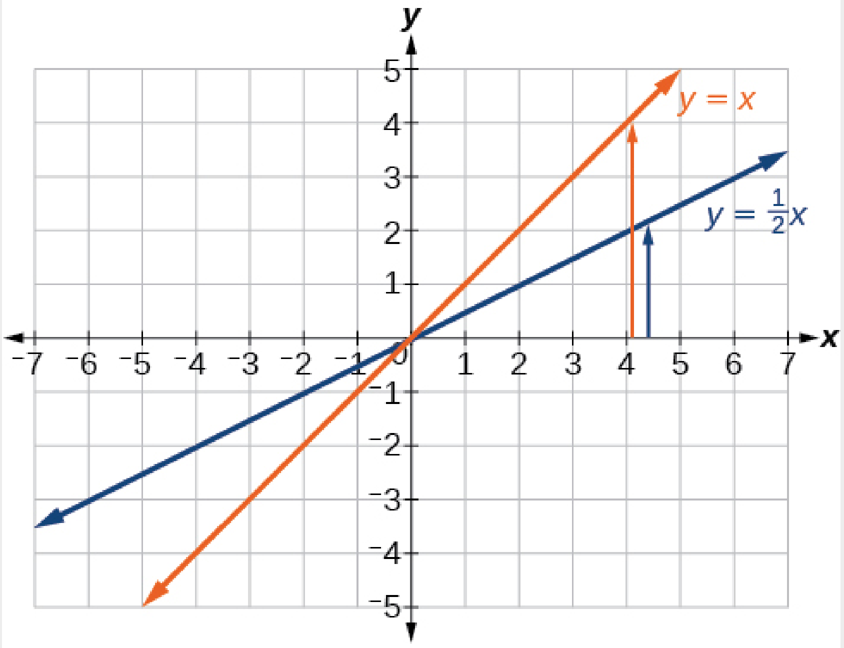
Problem:

Graphing by Using Transformations

Graph $f(x)=\frac{1}{2}x-3$ using transformations.

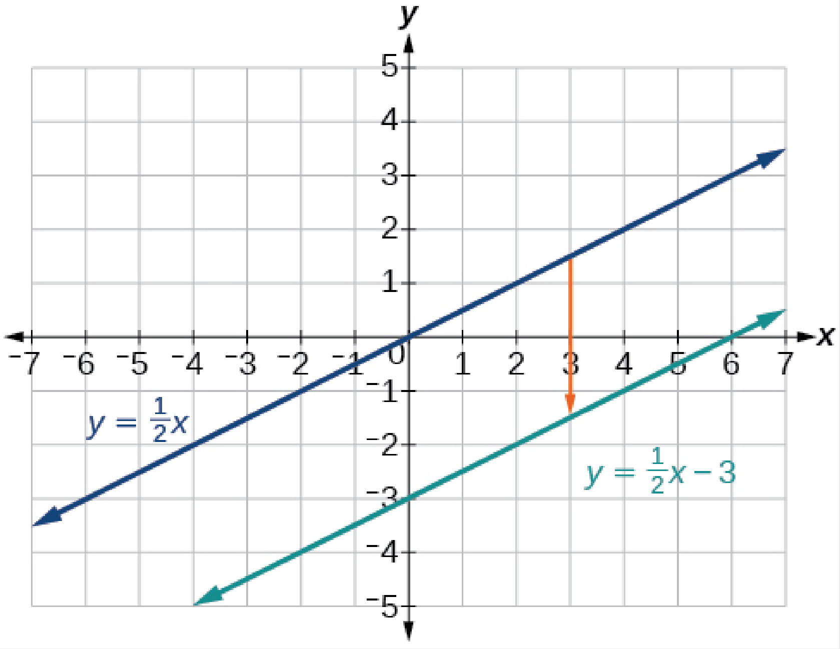
Solution:

The equation for the function shows that $m=\frac{1}{2}$ so the identity function is vertically compressed by $\frac{1}{2}$. The equation for the function also shows that $b=-3$ so the identity function is vertically shifted down 3 units. First, graph the identity function, and show the vertical compression as in [link].



The function, $y=x$,compressed by a factor of $\frac{1}{2}$

Then show the vertical shift as in [link].

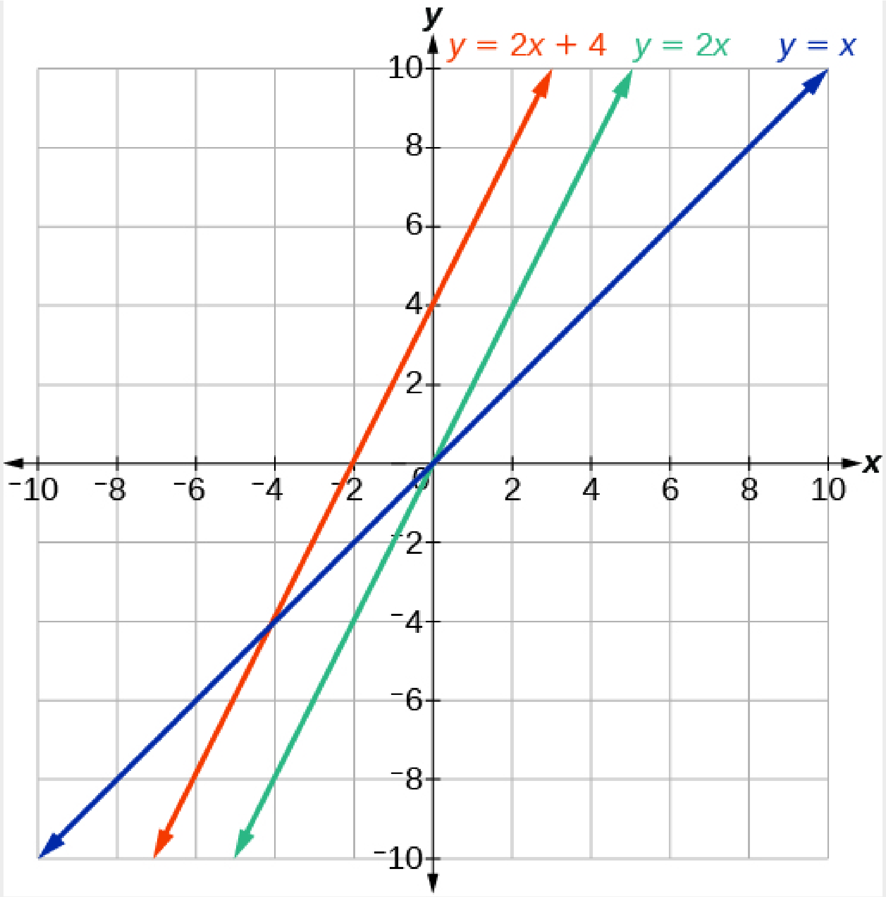


The function $y = \frac{1}{2}x$,shifted down 3 units

Note:
Exercise:

Problem: Graph $f(x) = 4 + 2x$ using transformations.

Solution:



Note:

In [link], could we have sketched the graph by reversing the order of the transformations?

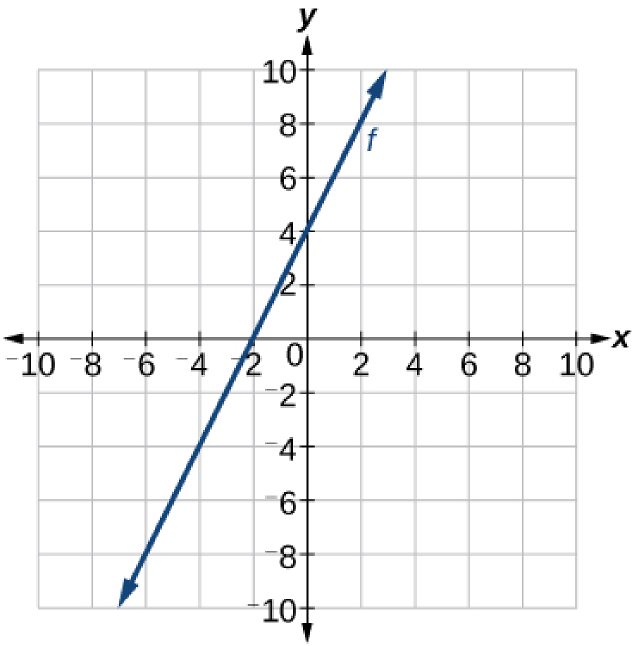
No. The order of the transformations follows the order of operations. When the function is evaluated at a given input, the corresponding output is calculated by following the order of operations. This is why we performed the compression first. For example, following the order: Let the input be 2.

Equation:

$$\begin{aligned} f(2) &= \tfrac{1}{2}(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Writing the Equation for a Function from the Graph of a Line

Earlier, we wrote the equation for a linear function from a graph. Now we can extend what we know about graphing linear functions to analyze graphs a little more closely. Begin by taking a look at [link]. We can see right away that the graph crosses the *y*-axis at the point $(0,4)$ so this is the *y*-intercept.



Then we can calculate the slope by finding the rise and run. We can choose any two points, but let’s look at the point $(–2,0)$. To get from this point to the *y*-intercept, we must move up 4 units (rise) and to the right 2 units (run). So the slope must be

Equation:

$$m=\frac{\text{rise}}{\text{run}}=\frac{4}{2}=2$$

Substituting the slope and *y*-intercept into the slope-intercept form of a line gives

Equation:

$$y=2x+4$$

Note:

Given a graph of linear function, find the equation to describe the function.

1. Identify the *y*-intercept of an equation.
2. Choose two points to determine the slope.
3. Substitute the *y*-intercept and slope into the slope-intercept form of a line.

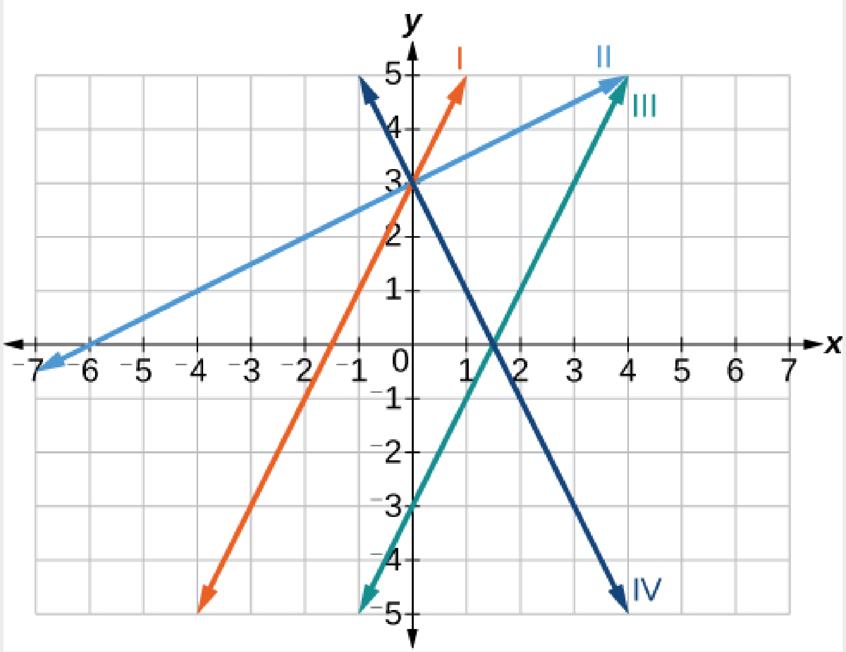
Example:
Exercise:

Problem:
Matching Linear Functions to Their Graphs

Match each equation of the linear functions with one of the lines in [link].

Equation:

$$\begin{array}{lccc} a. & f(x) & = & 2x+3 \\ b. & g(x) & = & 2x-3 \\ c. & h(x) & = & -2x+3 \\ d. & j(x) & = & \frac{1}{2}x+3 \end{array}$$

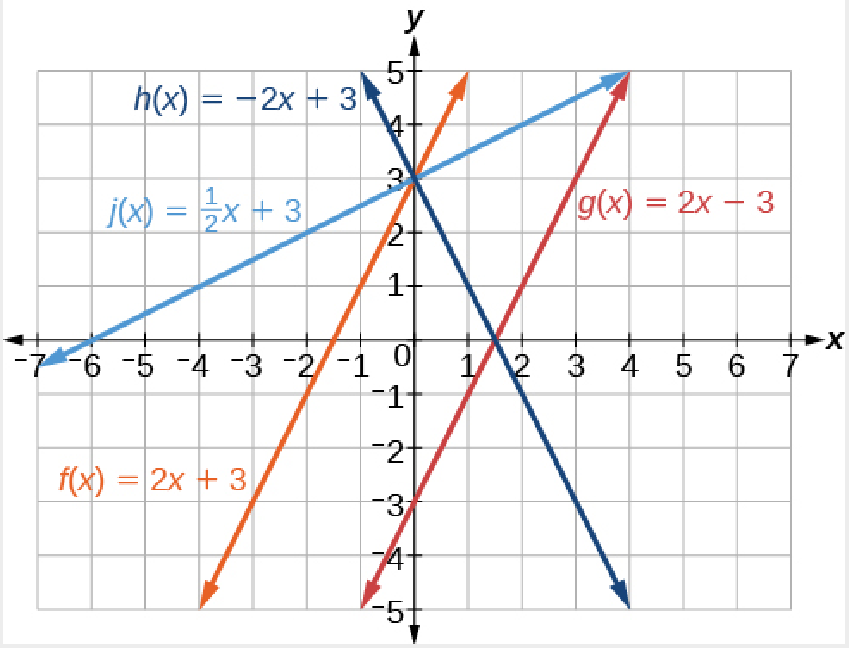


Solution:

Analyze the information for each function.

a. This function has a slope of 2 and a *y*-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. We can use two points to find the slope, or we can compare it with the other functions listed. Function g has the same slope, but a different *y*-intercept. Lines I and III have the same slant because they have the same slope. Line III does not pass through $(0, 3)$ so f must be represented by line I.
b. This function also has a slope of 2, but a *y*-intercept of -3. It must pass through the point $(0, -3)$ and slant upward from left to right. It must be represented by line III.
c. This function has a slope of –2 and a *y*-intercept of 3. This is the only function listed with a negative slope, so it must be represented by line IV because it slants downward from left to right.
d. This function has a slope of $\frac{1}{2}$ and a *y*-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. Lines I and II pass through $(0, 3)$,but the slope of j is less than the slope of f so the line for j must be flatter. This function is represented by Line II.

Now we can re-label the lines as in [link].



Finding the *x*-intercept of a Line

So far we have been finding the *y*-intercepts of a function: the point at which the graph of the function crosses the *y*-axis. Recall that a function may also have an *x*-intercept, which is the *x*-coordinate of the point where the graph of the function crosses the *x*-axis. In other words, it is the input value when the output value is zero.

To find the *x*-intercept, set a function $f(x)$ equal to zero and solve for the value of x. For example, consider the function shown.

Equation:

$$f(x) = 3x - 6$$

Set the function equal to 0 and solve for x.

Equation:

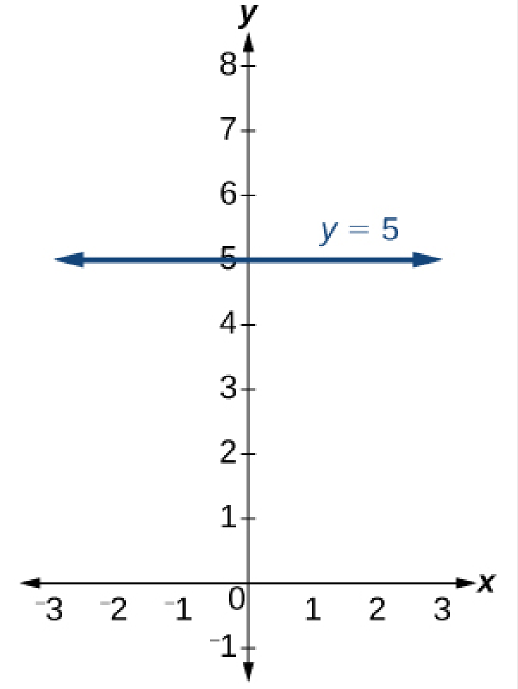
$$\begin{aligned} 0 &= 3x - 6 \\ 6 &= 3x \\ 2 &= x \\ x &= 2 \end{aligned}$$

The graph of the function crosses the *x*-axis at the point $(2, 0)$.

Note:

Do all linear functions have *x*-intercepts?

No. However, linear functions of the form $y = c$*,where* c *is a nonzero real number are the only examples of linear functions with no x-intercept. For example,* $y = 5$ *is a horizontal line 5 units above the x-axis. This function has no x-intercepts, as shown in* ***[link]****.*



Note:
x-intercept
The *x*-intercept of the function is value of x when $f(x) = 0$. It can be solved by the equation $0 = mx + b$.

Example:
Exercise:

Problem:
Finding an *x*-intercept

Find the *x*-intercept of $f(x) = \frac{1}{2}x - 3$.

Solution:

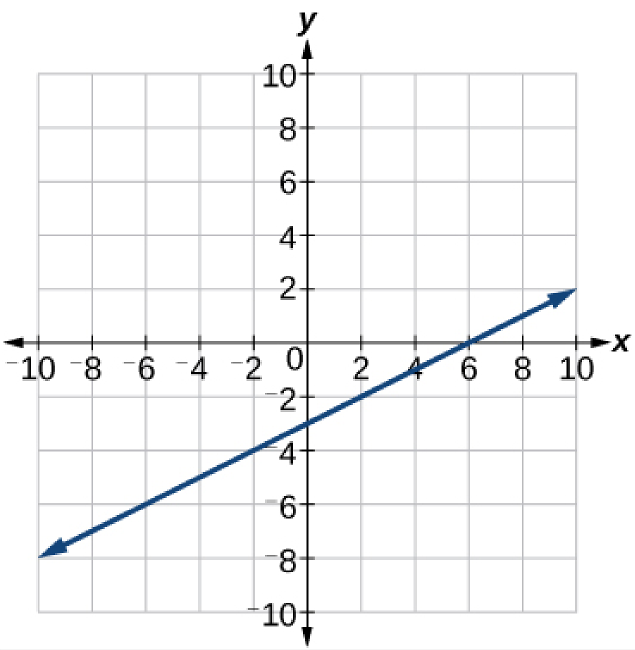
Set the function equal to zero to solve for x.
Equation:

$$\begin{aligned} 0 &= \tfrac{1}{2}x - 3 \\ 3 &= \tfrac{1}{2}x \\ 6 &= x \\ x &= 6 \end{aligned}$$

The graph crosses the *x*-axis at the point $(6, 0)$.

Analysis

A graph of the function is shown in [link]. We can see that the *x*-intercept is $(6, 0)$ as we expected.



Note:
Exercise:

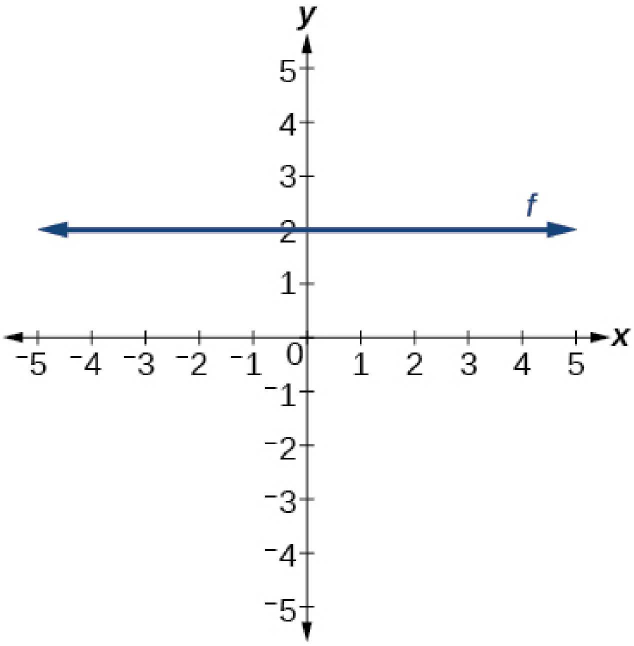
Problem: Find the *x*-intercept of $f(x) = \frac{1}{4}x - 4$.

Solution:

$(16,\ 0)$

Describing Horizontal and Vertical Lines

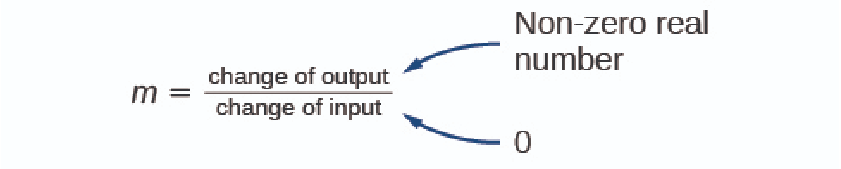
There are two special cases of lines on a graph—horizontal and vertical lines. A horizontal line indicates a constant output, or y-value. In [link], we see that the output has a value of 2 for every input value. The change in outputs between any two points, therefore, is 0. In the slope formula, the numerator is 0, so the slope is 0. If we use $m=0$ in the equation $f(x)=mx+b$,the equation simplifies to $f(x)=b$. In other words, the value of the function is a constant. This graph represents the function $f(x)=2$.



x	−4	−2	0	2	4
y	2	2	2	2	2

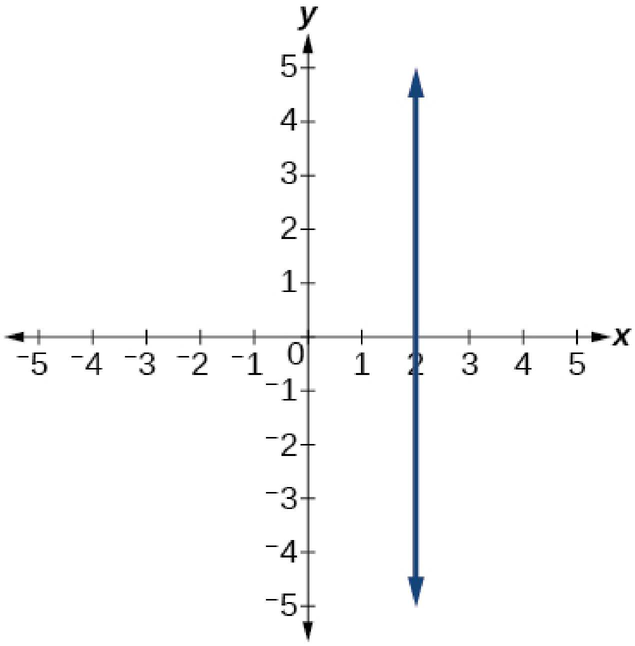
A horizontal line representing the function $f(x)=2$

A vertical line indicates a constant input, or x-value. We can see that the input value for every point on the line is 2, but the output value varies. Because this input value is mapped to more than one output value, a vertical line does not represent a function. Notice that between any two points, the change in the input values is zero. In the slope formula, the denominator will be zero, so the slope of a vertical line is undefined.


$m = \frac{\text{change of output}}{\text{change of input}}$
Non-zero real number
0

Example of how a line has a vertical slope. 0 in the denominator of the slope.

A vertical line, such as the one in [link], has an *x*-intercept, but no *y*-intercept unless it's the line $x = 0$. This graph represents the line $x = 2$.



x	2	2	2	2	2
y	−4	−2	0	2	4

The vertical line, $x = 2$, which does not represent a function

Note:
Horizontal and Vertical Lines

Lines can be horizontal or vertical.

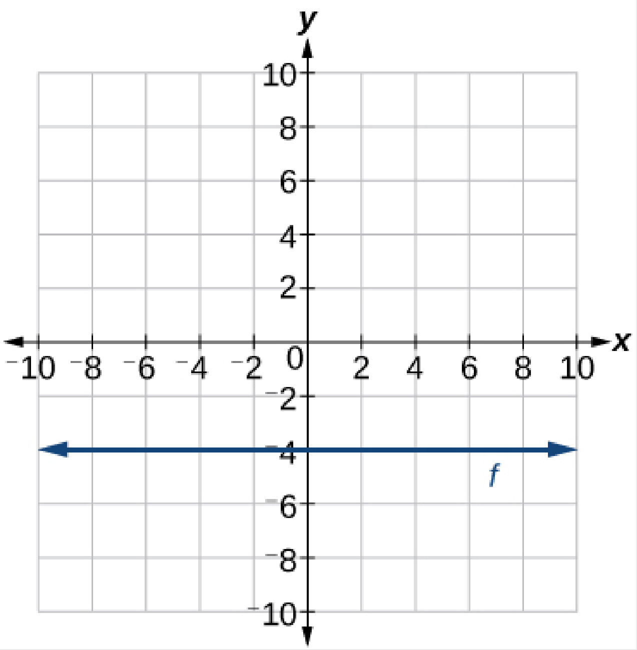
A **horizontal line** is a line defined by an equation in the form $f(x) = b$.

A **vertical line** is a line defined by an equation in the form $x = a$.

Example:
Exercise:

Problem:
Writing the Equation of a Horizontal Line

Write the equation of the line graphed in [link].



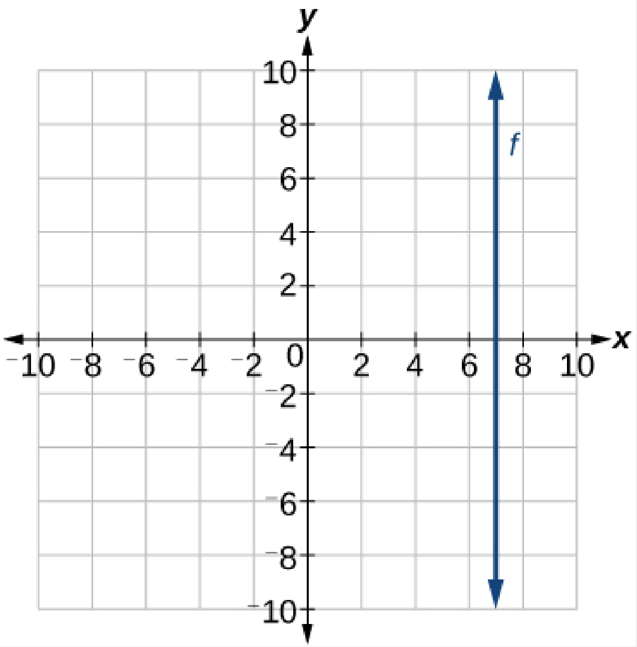
Solution:

For any *x*-value, the *y*-value is $-4,$ so the equation is $y = -4.$

Example:
Exercise:

Problem:
Writing the Equation of a Vertical Line

Write the equation of the line graphed in [link].

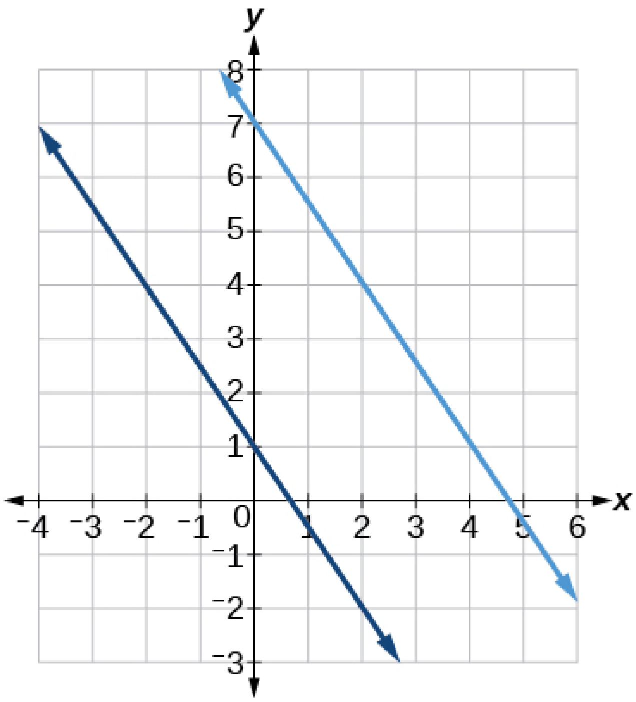


Solution:

The constant x-value is 7,so the equation is $x = 7$.

Determining Whether Lines are Parallel or Perpendicular

The two lines in [link] are parallel lines: they will never intersect. They have exactly the same steepness, which means their slopes are identical. The only difference between the two lines is the *y*-intercept. If we shifted one line vertically toward the other, they would become coincident.



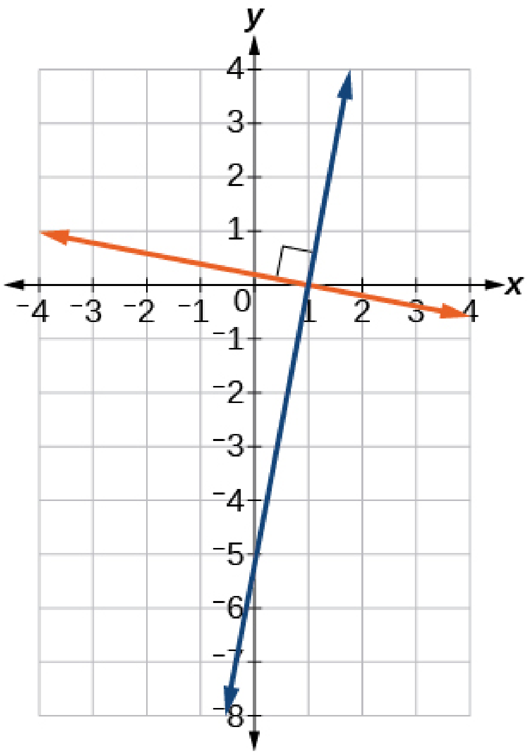
Parallel lines

We can determine from their equations whether two lines are parallel by comparing their slopes. If the slopes are the same and the *y*-intercepts are different, the lines are parallel. If the slopes are different, the lines are not parallel.

Equation:

$$\left.\begin{array}{l} f(x) = -2x + 6 \\ f(x) = -2x - 4 \end{array}\right\} \text{parallel} \qquad \left.\begin{array}{l} f(x) = 3x + 2 \\ f(x) = 2x + 2 \end{array}\right\} \text{not parallel}$$

Unlike parallel lines, perpendicular lines do intersect. Their intersection forms a right, or 90-degree, angle. The two lines in [link] are perpendicular.



Perpendicular lines

Perpendicular lines do not have the same slope. The slopes of perpendicular lines are different from one another in a specific way. The slope of one line is the negative reciprocal of the slope of the other line. The product of a number and its reciprocal is 1. So, if m_1 and m_2 are negative reciprocals of one another, they can be multiplied together to yield -1.

Equation:

$$m_1m_2 = -1$$

To find the reciprocal of a number, divide 1 by the number. So the reciprocal of 8 is $\frac{1}{8}$, and the reciprocal of $\frac{1}{8}$ is 8. To find the negative reciprocal, first find the reciprocal and then change the sign.

As with parallel lines, we can determine whether two lines are perpendicular by comparing their slopes, assuming that the lines are neither horizontal nor vertical. The slope of each line below is the negative reciprocal of the other so the lines are perpendicular.

Equation:

$$\begin{aligned} f(x) &= \tfrac{1}{4}x + 2 && \text{negative reciprocal of } \tfrac{1}{4} \text{ is } -4 \\ f(x) &= -4x + 3 && \text{negative reciprocal of } -4 \text{ is } \tfrac{1}{4} \end{aligned}$$

The product of the slopes is –1.

Equation:

$$-4\left(\frac{1}{4}\right) = -1$$

Note:

Parallel and Perpendicular Lines

Two lines are **parallel lines** if they do not intersect. The slopes of the lines are the same.

Equation:

$$f(x) = m_1x + b_1 \text{ and } g(x) = m_2x + b_2 \text{ are parallel if and only if } m_1 = m_2$$

If and only if $b_1 = b_2$ and $m_1 = m_2,$ we say the lines coincide. Coincident lines are the same line.

Two lines are **perpendicular lines** if they intersect to form a right angle.

Equation:

$$f(x) = m_1x + b_1 \text{ and } g(x) = m_2x + b_2 \text{ are perpendicular if and only if}$$

Equation:

$$m_1m_2 = -1, \text{so } m_2 = -\frac{1}{m_1}$$

Example:

Exercise:

Problem:

Identifying Parallel and Perpendicular Lines

Given the functions below, identify the functions whose graphs are a pair of parallel lines and a pair of perpendicular lines.

Equation:

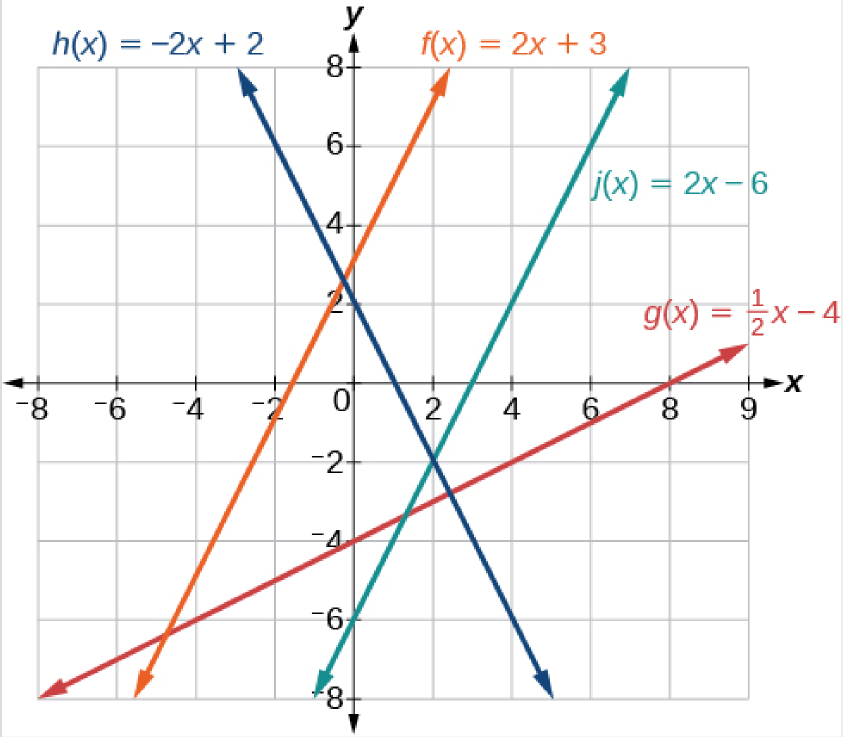
$$\begin{array}{rclcrcl} f(x) & = & 2x+3 & \qquad & h(x) & = & -2x+2 \\ g(x) & = & \frac{1}{2}x-4 & \qquad & j(x) & = & 2x-6 \end{array}$$

Solution:

Parallel lines have the same slope. Because the functions $f(x)=2x+3$ and $j(x)=2x-6$ each have a slope of 2, they represent parallel lines. Perpendicular lines have negative reciprocal slopes. Because −2 and $\frac{1}{2}$ are negative reciprocals, the functions $g(x)=\frac{1}{2}x-4$ and $h(x)=-2x+2$ represent perpendicular lines.

Analysis

A graph of the lines is shown in [link].



The graph shows that the lines $f(x)=2x+3$ and $j(x)=2x-6$ are parallel, and the lines $g(x)=\frac{1}{2}x-4$ and $h(x)=-2x+2$ are perpendicular.

Writing the Equation of a Line Parallel or Perpendicular to a Given Line

If we know the equation of a line, we can use what we know about slope to write the equation of a line that is either parallel or perpendicular to the given line.

Writing Equations of Parallel Lines

Suppose for example, we are given the equation shown.

Equation:

$$f(x) = 3x + 1$$

We know that the slope of the line formed by the function is 3. We also know that the *y*-intercept is $(0, 1).$ Any other line with a slope of 3 will be parallel to $f(x).$ So the lines formed by all of the following functions will be parallel to $f(x).$

Equation:

$$\begin{array}{ccc} g(x) & = & 3x + 6 \\ h(x) & = & 3x + 1 \\ p(x) & = & 3x + \frac{2}{3} \end{array}$$

Suppose then we want to write the equation of a line that is parallel to f and passes through the point $(1, 7).$ This type of problem is often described as a point-slope problem because we have a point and a slope. In our example, we know that the slope is 3. We need to determine which value of b will give the correct line. We can begin with the point-slope form of an equation for a line, and then rewrite it in the slope-intercept form.

Equation:

$$\begin{array}{rcl} y - y_1 & = & m(x - x_1) \\ y - 7 & = & 3(x - 1) \\ y - 7 & = & 3x - 3 \\ y & = & 3x + 4 \end{array}$$

So $g(x) = 3x + 4$ is parallel to $f\,(x) = 3x + 1$ and passes through the point $(1, 7).$

Note:

Given the equation of a function and a point through which its graph passes, write the equation of a line parallel to the given line that passes through the given point.

1. Find the slope of the function.
2. Substitute the given values into either the general point-slope equation or the slope-intercept equation for a line.

3. Simplify.

Example:
Exercise:

Problem:
Finding a Line Parallel to a Given Line

Find a line parallel to the graph of $f(x) = 3x + 6$ that passes through the point $(3, 0)$.

Solution:

The slope of the given line is 3. If we choose the slope-intercept form, we can substitute $m = 3$, $x = 3$, and $f(x) = 0$ into the slope-intercept form to find the *y*-intercept.

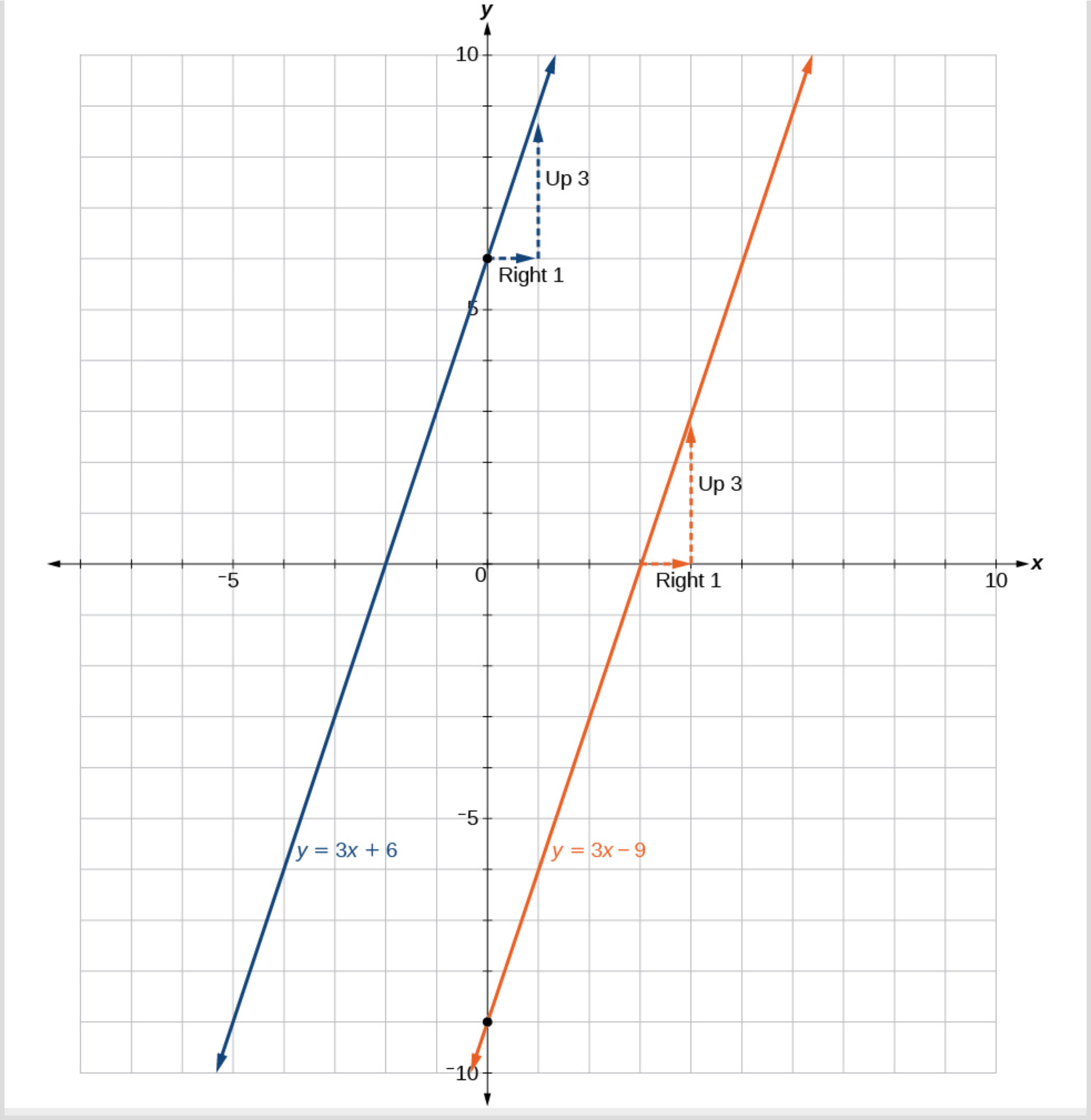
Equation:

$$\begin{aligned} g(x) &= 3x + b \\ 0 &= 3(3) + b \\ b &= -9 \end{aligned}$$

The line parallel to $f(x)$ that passes through $(3, 0)$ is $g(x) = 3x - 9$.

Analysis

We can confirm that the two lines are parallel by graphing them. [link] shows that the two lines will never intersect.



Writing Equations of Perpendicular Lines

We can use a very similar process to write the equation for a line perpendicular to a given line. Instead of using the same slope, however, we use the negative reciprocal of the given slope. Suppose we are given the function shown.

Equation:

$$f(x) = 2x + 4$$

The slope of the line is 2, and its negative reciprocal is $-\frac{1}{2}.$ Any function with a slope of $-\frac{1}{2}$ will be perpendicular to $f(x).$ So the lines formed by all of the following functions will be perpendicular to $f(x).$

Equation:

$$\begin{array}{rcl} g(x) & = & -\frac{1}{2}x+4 \\ h(x) & = & -\frac{1}{2}x+2 \\ p(x) & = & -\frac{1}{2}x-\frac{1}{2} \end{array}$$

As before, we can narrow down our choices for a particular perpendicular line if we know that it passes through a given point. Suppose then we want to write the equation of a line that is perpendicular to $f(x)$ and passes through the point $(4,0).$ We already know that the slope is $-\frac{1}{2}.$ Now we can use the point to find the *y*-intercept by substituting the given values into the slope-intercept form of a line and solving for b.

Equation:

$$\begin{array}{rcl} g(x) & = & mx+b \\ 0 & = & -\frac{1}{2}(4)+b \\ 0 & = & -2+b \\ 2 & = & b \\ b & = & 2 \end{array}$$

The equation for the function with a slope of $-\frac{1}{2}$ and a *y*-intercept of 2 is

Equation:

$$g(x)=-\frac{1}{2}x+2$$

So $g(x)=-\frac{1}{2}x+2$ is perpendicular to $f\left(x\right)=2x+4$ and passes through the point $(4,0).$ Be aware that perpendicular lines may not look obviously perpendicular on a graphing calculator unless we use the square zoom feature.

Note:

A horizontal line has a slope of zero and a vertical line has an undefined slope. These two lines are perpendicular, but the product of their slopes is not –1. Doesn’t this fact contradict the definition of perpendicular lines?

No. For two perpendicular linear functions, the product of their slopes is –1. However, a vertical line is not a function so the definition is not contradicted.

Note:
Given the equation of a function and a point through which its graph passes, write the equation of a line perpendicular to the given line.

1. Find the slope of the function.
2. Determine the negative reciprocal of the slope.
3. Substitute the new slope and the values for x and y from the coordinate pair provided into $g\left(x\right)=mx+b$.
4. Solve for b.
5. Write the equation of the line.

Example:
Exercise:

Problem:
Finding the Equation of a Perpendicular Line

Find the equation of a line perpendicular to $f(x)=3x+3$ that passes through the point $(3,0)$.

Solution:

The original line has slope $m=3,$ so the slope of the perpendicular line will be its negative reciprocal, or $-\frac{1}{3}$. Using this slope and the given point, we can find the equation of the line.

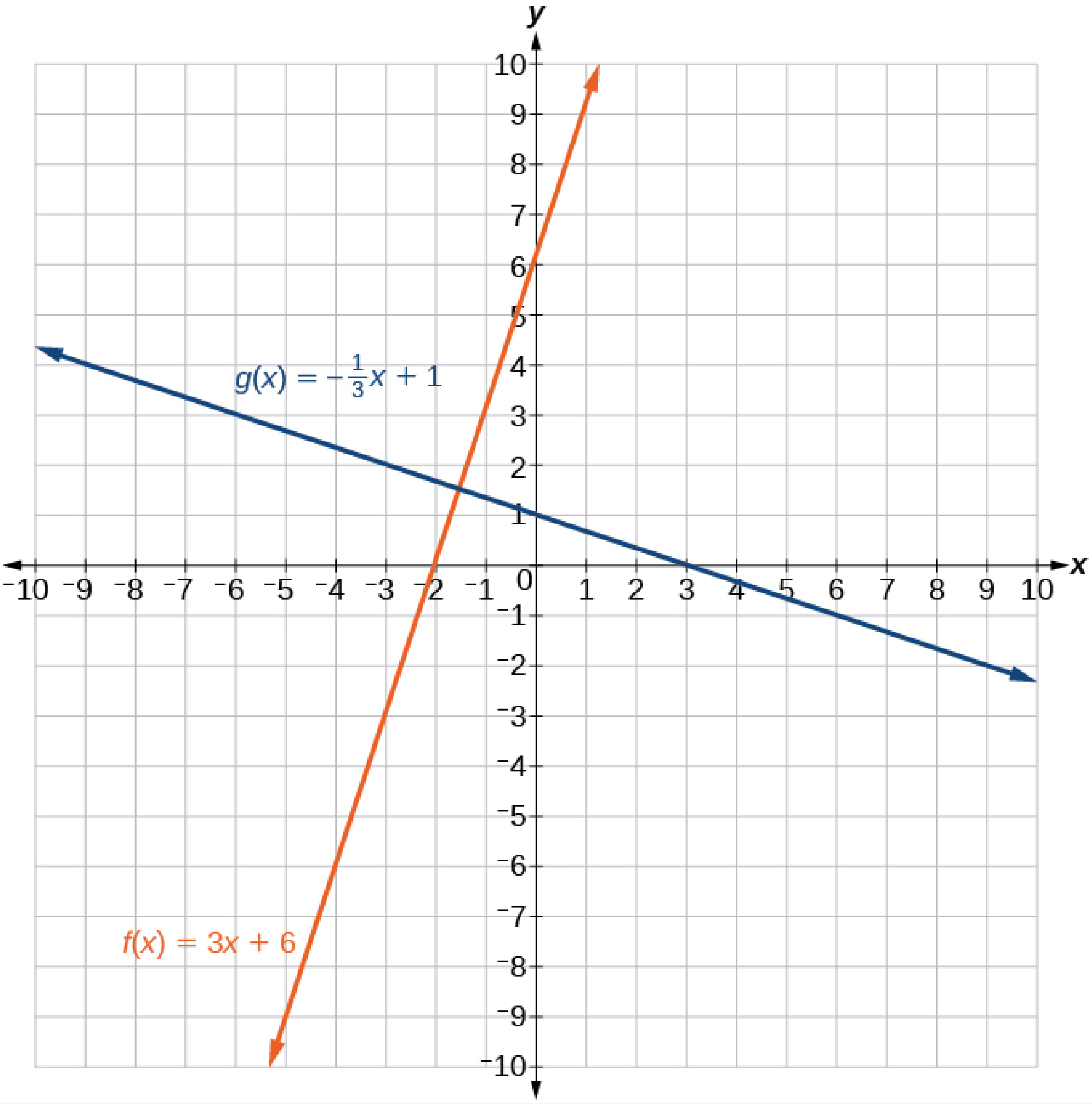
Equation:

$$\begin{array}{rcl} g(x) & = & -\frac{1}{3}x+b \\ 0 & = & -\frac{1}{3}(3)+b \\ 1 & = & b \\ b & = & 1 \end{array}$$

The line perpendicular to $f(x)$ that passes through $(3,0)$ is $g(x)=-\frac{1}{3}x+1$.

Analysis

A graph of the two lines is shown in [link].



Note that that if we graph perpendicular lines on a graphing calculator using standard zoom, the lines may not appear to be perpendicular. Adjusting the window will make it possible to zoom in further to see the intersection more closely.

Note:
Exercise:

Problem:

Given the function $h(x) = 2x - 4$,write an equation for the line passing through $(0, 0)$ that is

a. parallel to $h(x)$
b. perpendicular to $h(x)$

Solution:

a. $f(x) = 2x$;b. $g(x) = -\frac{1}{2}x$

Note:
Given two points on a line and a third point, write the equation of the perpendicular line that passes through the point.

1. Determine the slope of the line passing through the points.
2. Find the negative reciprocal of the slope.
3. Use the slope-intercept form or point-slope form to write the equation by substituting the known values.
4. Simplify.

Example:
Exercise:

Problem:
Finding the Equation of a Line Perpendicular to a Given Line Passing through a Point

A line passes through the points $(-2, 6)$ and $(4, 5)$. Find the equation of a perpendicular line that passes through the point $(4, 5)$.

Solution:

From the two points of the given line, we can calculate the slope of that line.
Equation:

$$\begin{aligned} m_1 &= \frac{5-6}{4-(-2)} \\ &= \frac{-1}{6} \\ &= -\frac{1}{6} \end{aligned}$$

Find the negative reciprocal of the slope.

Equation:

$$\begin{aligned} m_2 &= \frac{-1}{-\frac{1}{6}} \\ &= -1\left(-\frac{6}{1}\right) \\ &= 6 \end{aligned}$$

We can then solve for the *y*-intercept of the line passing through the point $(4,5)$.

Equation:

$$\begin{aligned} g(x) &= 6x+b \\ 5 &= 6(4)+b \\ 5 &= 24+b \\ -19 &= b \\ b &= -19 \end{aligned}$$

The equation for the line that is perpendicular to the line passing through the two given points and also passes through point $(4,5)$ is

Equation:

$$y=6x-19$$

Note:

Exercise:

Problem:

A line passes through the points, $(-2,-15)$ and $(2,-3)$. Find the equation of a perpendicular line that passes through the point, $(6,4)$.

Solution:

$y=-\frac{1}{3}x+6$

Note:

Access this online resource for additional instruction and practice with linear functions.

- Linear Functions
- Finding Input of Function from the Output and Graph
- Graphing Functions using Tables

Key Concepts

- Linear functions can be represented in words, function notation, tabular form, and graphical form. See [link].
- An increasing linear function results in a graph that slants upward from left to right and has a positive slope. A decreasing linear function results in a graph that slants downward from left to right and has a negative slope. A constant linear function results in a graph that is a horizontal line. See [link].
- Slope is a rate of change. The slope of a linear function can be calculated by dividing the difference between *y*-values by the difference in corresponding *x*-values of any two points on the line. See [link] and [link].
- An equation for a linear function can be written from a graph. See [link].
- The equation for a linear function can be written if the slope m and initial value b are known. See [link] and [link].
- A linear function can be used to solve real-world problems given information in different forms. See [link], [link], and [link].
- Linear functions can be graphed by plotting points or by using the *y*-intercept and slope. See [link] and [link].
- Graphs of linear functions may be transformed by using shifts up, down, left, or right, as well as through stretches, compressions, and reflections. See [link].
- The equation for a linear function can be written by interpreting the graph. See [link].
- The *x*-intercept is the point at which the graph of a linear function crosses the *x*-axis. See [link].
- Horizontal lines are written in the form, $f(x) = b.$ See [link].
- Vertical lines are written in the form, $x = b.$ See [link].
- Parallel lines have the same slope. Perpendicular lines have negative reciprocal slopes, assuming neither is vertical. See [link].
- A line parallel to another line, passing through a given point, may be found by substituting the slope value of the line and the *x*- and *y*-values of the given point into the equation, $f(x) = mx + b,$ and using the b that results. Similarly, the point-slope form of an equation can also be used. See [link].
- A line perpendicular to another line, passing through a given point, may be found in the same manner, with the exception of using the negative reciprocal slope. See

[link] and [link].

Section Exercises

Verbal

Exercise:

Problem:

Terry is skiing down a steep hill. Terry's elevation, $E(t)$,in feet after t seconds is given by $E(t) = 3000 - 70t.$ Write a complete sentence describing Terry’s starting elevation and how it is changing over time.

Solution:

Terry starts at an elevation of 3000 feet and descends 70 feet per second.

Exercise:

Problem:

Jessica is walking home from a friend’s house. After 2 minutes she is 1.4 miles from home. Twelve minutes after leaving, she is 0.9 miles from home. What is her rate in miles per hour?

Exercise:

Problem:

A boat is 100 miles away from the marina, sailing directly toward it at 10 miles per hour. Write an equation for the distance of the boat from the marina after t hours.

Solution:

$d\left(t\right) = 100 - 10t$

Exercise:

Problem:

If the graphs of two linear functions are perpendicular, describe the relationship between the slopes and the y-intercepts.

Exercise:

Problem:

If a horizontal line has the equation $f\left(x\right)=a$ and a vertical line has the equation $x=a,$ what is the point of intersection? Explain why what you found is the point of intersection.

Solution:

The point of intersection is $\left(a,\text{ }a\right).$ This is because for the horizontal line, all of the y coordinates are a and for the vertical line, all of the x coordinates are $a.$ The point of intersection is on both lines and therefore will have these two characteristics.

Algebraic

For the following exercises, determine whether the equation of the curve can be written as a linear function.

Exercise:

Problem:$y=\frac{1}{4}x+6$

Exercise:

Problem:$y=3x-5$

Solution:

Yes

Exercise:

Problem:$y=3x^{2}-2$

Exercise:

Problem:$3x+5y=15$

Solution:

Yes

Exercise:

Problem: $3x^2 + 5y = 15$

Exercise:

Problem: $3x + 5y^2 = 15$

Solution:

No

Exercise:

Problem: $-2x^2 + 3y^2 = 6$

Exercise:

Problem: $-\frac{x-3}{5} = 2y$

Solution:

Yes

For the following exercises, determine whether each function is increasing or decreasing.

Exercise:

Problem: $f\left(x\right) = 4x + 3$

Exercise:

Problem: $g\left(x\right) = 5x + 6$

Solution:

Increasing

Exercise:

Problem: $a\left(x\right) = 5 - 2x$

Exercise:

Problem: $b\left(x\right) = 8 - 3x$

Solution:

Decreasing

Exercise:

Problem:$h\left(x\right)=-2x+4$

Exercise:

Problem:$k\left(x\right)=-4x+1$

Solution:

Decreasing

Exercise:

Problem:$j\left(x\right)=\frac{1}{2}x-3$

Exercise:

Problem:$p\left(x\right)=\frac{1}{4}x-5$

Solution:

Increasing

Exercise:

Problem:$n\left(x\right)=-\frac{1}{3}x-2$

Exercise:

Problem:$m\left(x\right)=-\frac{3}{8}x+3$

Solution:

Decreasing

For the following exercises, find the slope of the line that passes through the two given points.

Exercise:

Problem: $(2, 4)$ and $(4, 10)$

Exercise:

Problem: $(1, 5)$ and $(4, 11)$

Solution:

2

Exercise:

Problem: $(–1, 4)$ and $(5, 2)$

Exercise:

Problem: $(8, –2)$ and $(4, 6)$

Solution:

–2

Exercise:

Problem: $(6, 11)$ and $(–4, 3)$

For the following exercises, given each set of information, find a linear equation satisfying the conditions, if possible.

Exercise:

Problem: $f(-5) = -4,$ and $f(5) = 2$

Solution:

$y = \frac{3}{5}x - 1$

Exercise:

Problem: $f(-1) = 4,$ and $f(5) = 1$

Exercise:

Problem: Passes through $(2, 4)$ and $(4, 10)$

Solution:

$y = 3x - 2$

Exercise:

Problem: Passes through $(1, 5)$ and $(4, 11)$

Exercise:

Problem: Passes through $(-1, 4)$ and $(5, 2)$

Solution:

$y = -\frac{1}{3}x + \frac{11}{3}$

Exercise:

Problem: Passes through $(-2, 8)$ and $(4, 6)$

Exercise:

Problem:*x* intercept at $(-2, 0)$ and *y* intercept at $(0, -3)$

Solution:

$y = -1.5x - 3$

Exercise:

Problem:*x* intercept at $(-5, 0)$ and *y* intercept at $(0, 4)$

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither.

Exercise:

Problem:
$$4x - 7y = 10$$
$$7x + 4y = 1$$

Solution:

perpendicular

Exercise:

Problem: $$\begin{array}{l}3y+x=12\\ -y=8x+1\end{array}$$

Exercise:

Problem: $$\begin{array}{l}3y+4x=12\\ -6y=8x+1\end{array}$$

Solution:

parallel

Exercise:

Problem: $$\begin{array}{l}6x-9y=10\\ 3x+2y=1\end{array}$$

For the following exercises, find the *x*- and *y*-intercepts of each equation.

Exercise:

Problem: $f\left(x\right)=-x+2$

Solution:

$$\begin{array}{l}f(0)=-(0)+2\\ f(0)=2\\ y-\text{int}:(0,2)\\ 0=-x+2\\ x-\text{int}:(2,0)\end{array}$$

Exercise:

Problem: $g\left(x\right)=2x+4$

Exercise:

Problem: $h\left(x\right)=3x-5$

Solution:

$h(0) = 3(0) - 5$
$h(0) = -5$
$y - \text{int} : (0, -5)$
$0 = 3x - 5$
$x - \text{int} : \left(\frac{5}{3}, 0\right)$

Exercise:

Problem: $k(x) = -5x + 1$

Exercise:

Problem: $-2x + 5y = 20$

Solution:

$-2x + 5y = 20$
$-2(0) + 5y = 20$
$5y = 20$
$y = 4$
$y - \text{int} : (0, 4)$
$-2x + 5(0) = 20$
$x = -10$
$x - \text{int} : (-10, 0)$

Exercise:

Problem: $7x + 2y = 56$

For the following exercises, use the descriptions of each pair of lines given below to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?

Exercise:

Problem: Line 1: Passes through $(0, 6)$ and $(3, -24)$

Line 2: Passes through $(-1, 19)$ and $(8, -71)$

Solution:

Line 1: *m* = –10 Line 2: *m* = –10 Parallel

Exercise:

Problem: Line 1: Passes through $(-8,-55)$ and $(10,89)$

Line 2: Passes through $(9,-44)$ and $(4,-14)$

Exercise:

Problem: Line 1: Passes through $(2,3)$ and $(4,-1)$

Line 2: Passes through $(6,3)$ and $(8,5)$

Solution:

Line 1: m = –2 Line 2: m = 1 Neither

Exercise:

Problem: Line 1: Passes through $(1,7)$ and $(5,5)$

Line 2: Passes through $(-1,-3)$ and $(1,1)$

Exercise:

Problem: Line 1: Passes through $(2,5)$ and $(5,-1)$

Line 2: Passes through $(-3,7)$ and $(3,-5)$

Solution:

$\text{Line 1}: \ m =-2 \quad \text{Line 2}: \ m =-2 \quad \text{Parallel}$

For the following exercises, write an equation for the line described.

Exercise:

Problem:

Write an equation for a line parallel to $f(x)=-5x-3$ and passing through the point $(2,–12)$.

Exercise:

Problem:

Write an equation for a line parallel to $g(x) = 3x - 1$ and passing through the point $(4, 9)$.

Solution:

$y = 3x - 3$

Exercise:

Problem:

Write an equation for a line perpendicular to $h(t) = -2t + 4$ and passing through the point $(-4, -1)$.

Exercise:

Problem:

Write an equation for a line perpendicular to $p(t) = 3t + 4$ and passing through the point $(3, 1)$.

Solution:

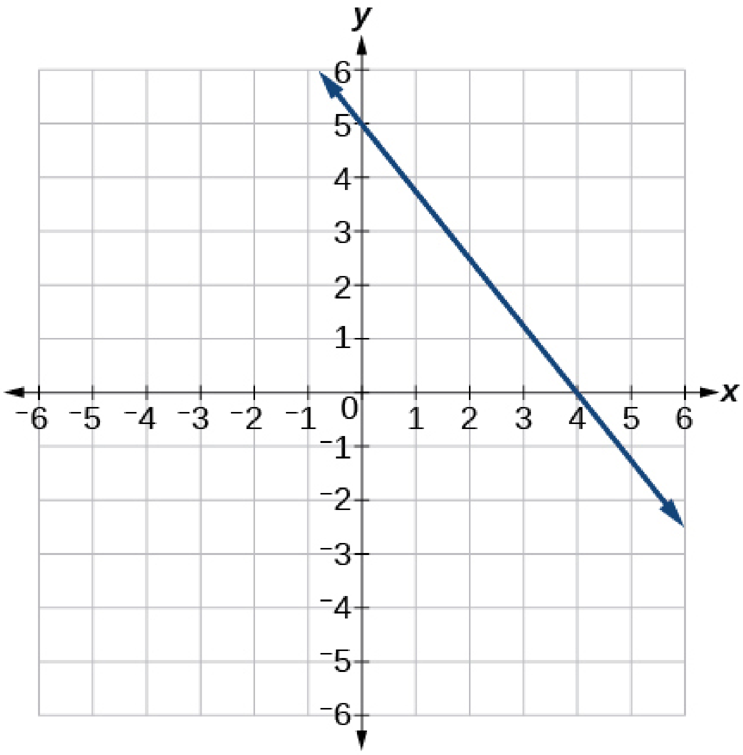
$y = -\frac{1}{3}t + 2$

Graphical

For the following exercises, find the slope of the line graphed.

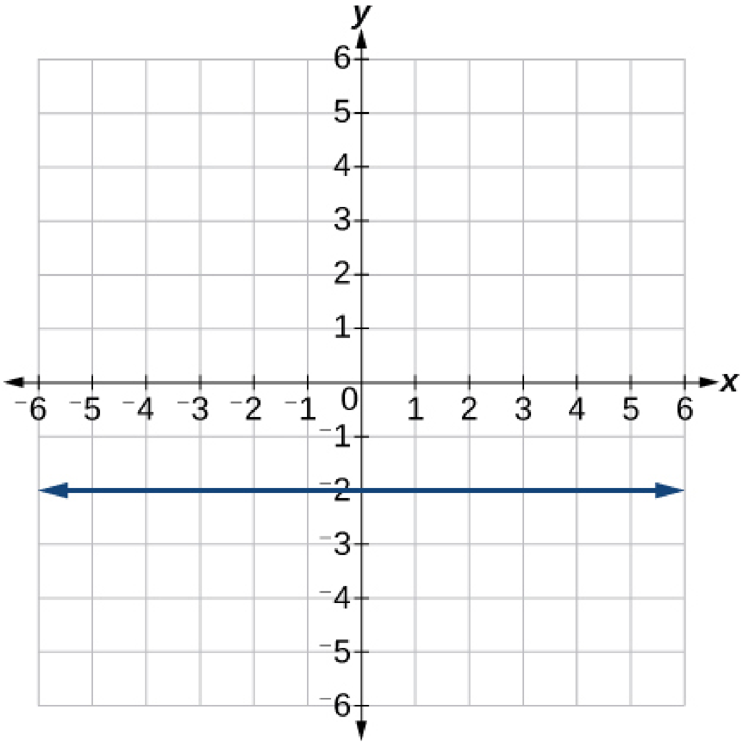
Exercise:

Problem:



Exercise:

Problem:



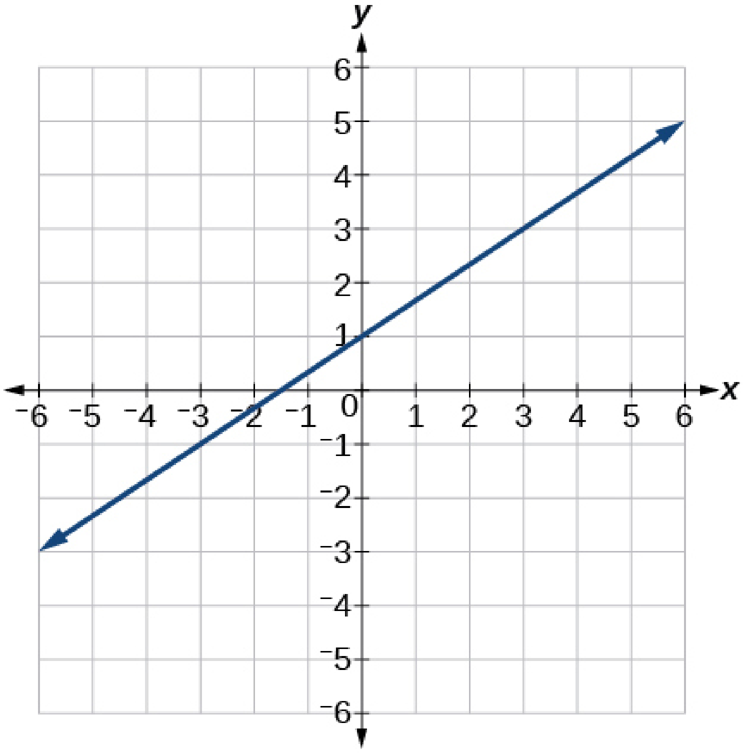
Solution:

0

For the following exercises, write an equation for the line graphed.

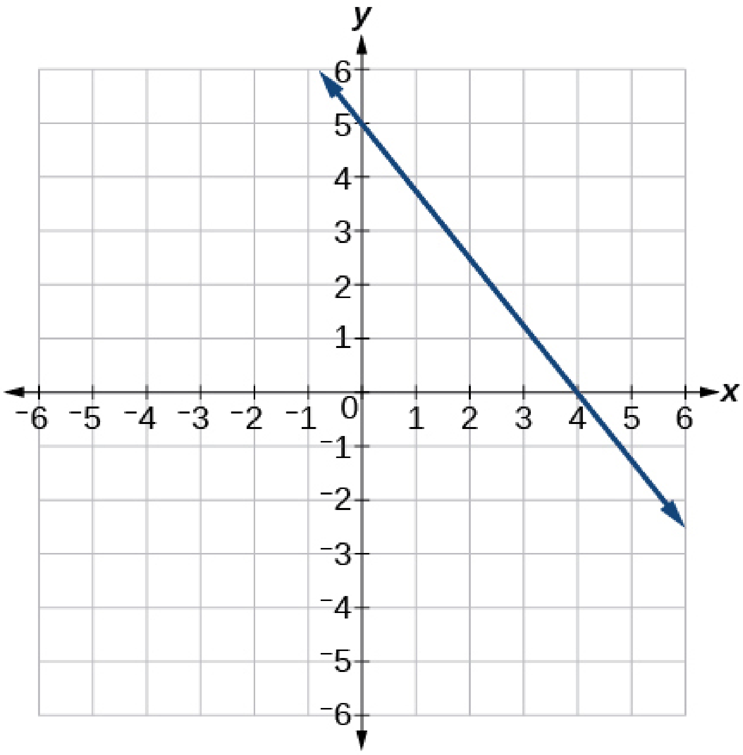
Exercise:

Problem:



Exercise:

Problem:

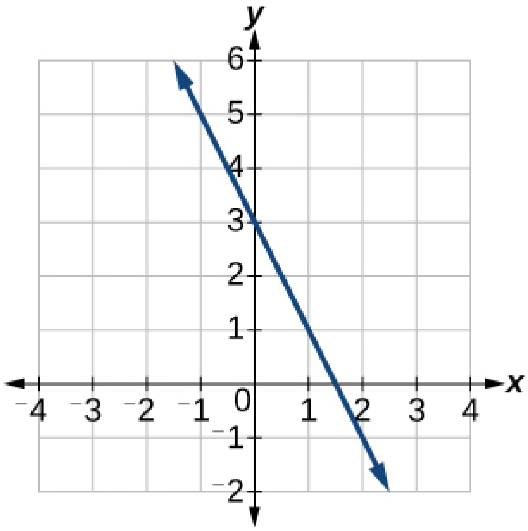


Solution:

$y = -\frac{5}{4}x + 5$

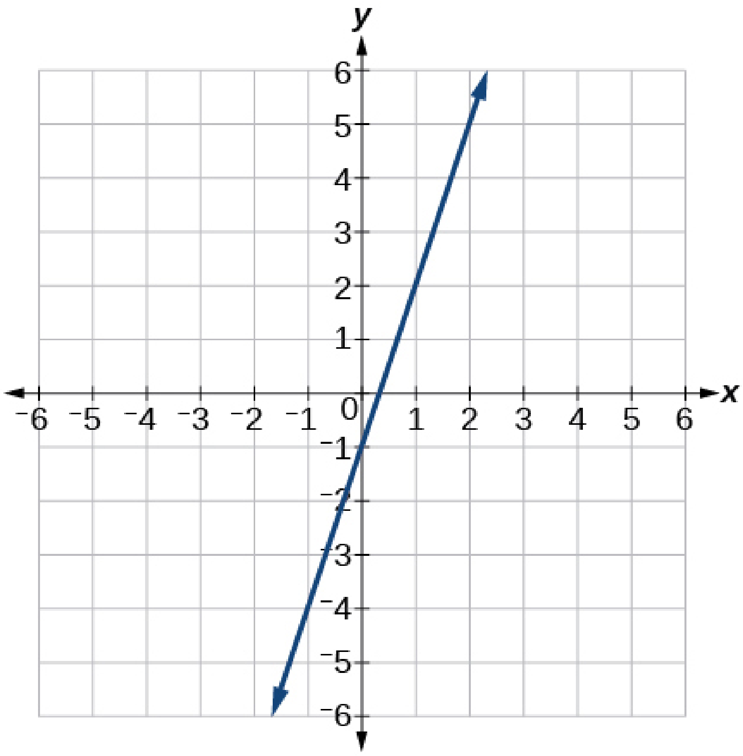
Exercise:

Problem:



Exercise:

Problem:

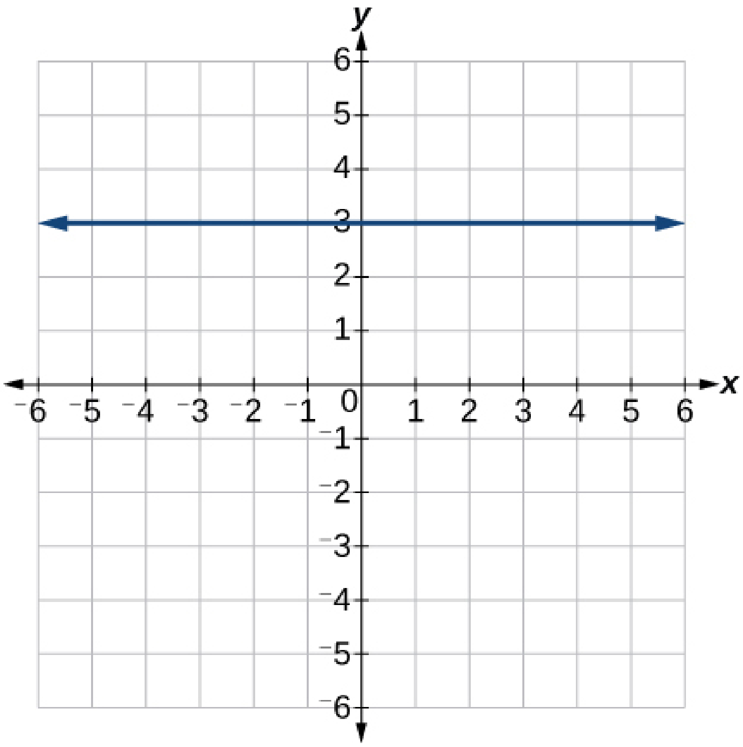


Solution:

$y = 3x - 1$

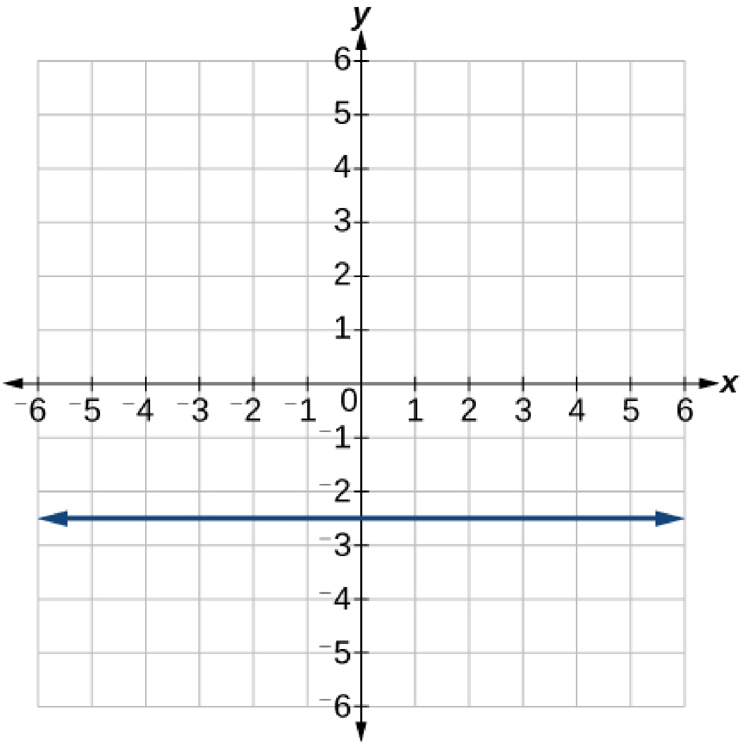
Exercise:

Problem:



Exercise:

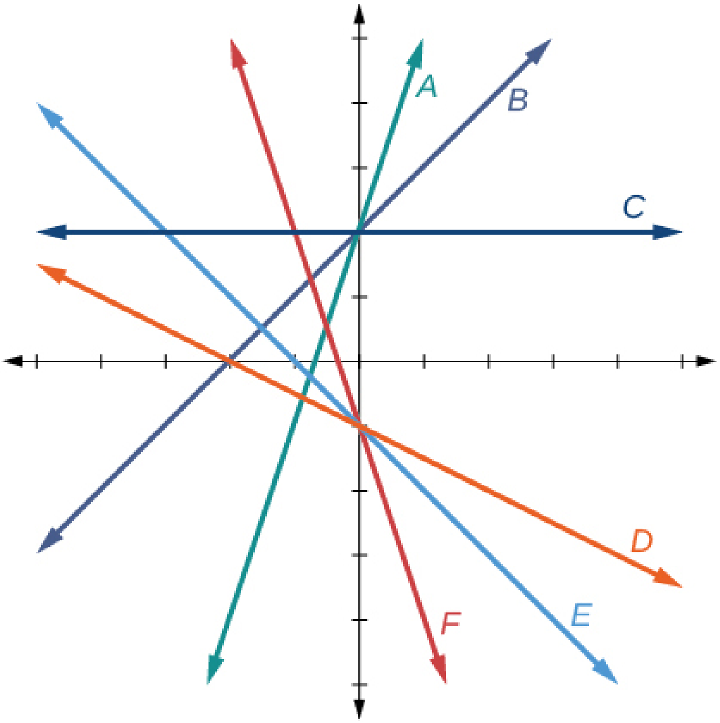
Problem:



Solution:

$y = -2.5$

For the following exercises, match the given linear equation with its graph in [link].



Exercise:

Problem: $f\left(x\right) = -x - 1$

Exercise:

Problem: $f\left(x\right) = -2x - 1$

Solution:

F

Exercise:

Problem: $f\left(x\right) = -\frac{1}{2}x - 1$

Exercise:

Problem: $f\left(x\right)=2$

Solution:

C

Exercise:

Problem: $f\left(x\right)=2+x$

Exercise:

Problem: $f\left(x\right)=3x+2$

Solution:

A

For the following exercises, sketch a line with the given features.

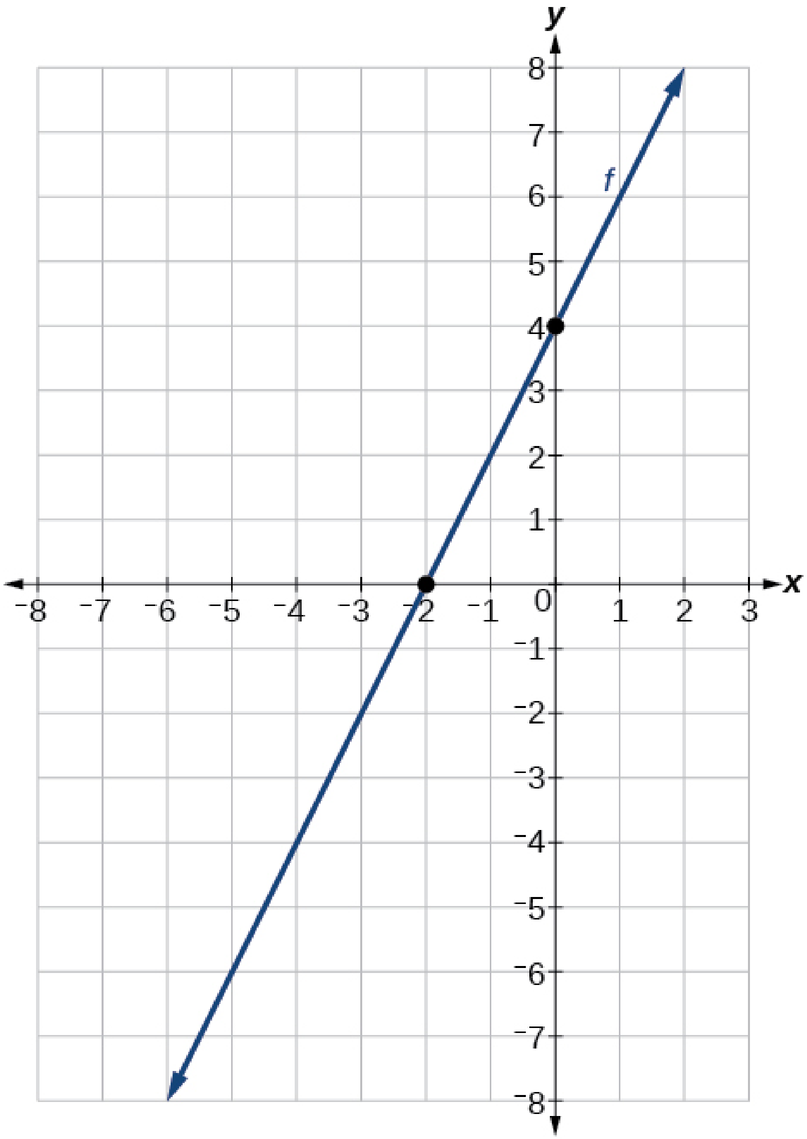
Exercise:

Problem: An *x*-intercept of $\left(–4,\text{ }0\right)$ and *y*-intercept of $\left(0,\text{ }–2\right)$

Exercise:

Problem: An *x*-intercept $\left(–2,\text{ }0\right)$ and *y*-intercept of $\left(0,\text{ }4\right)$

Solution:



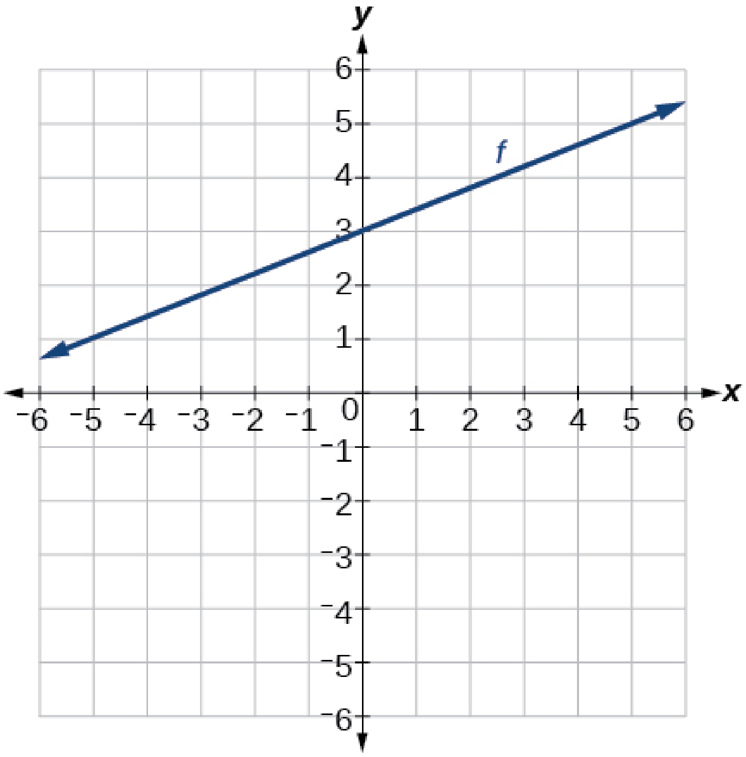
Exercise:

Problem: A y-intercept of $(0,7)$ and slope $-\frac{3}{2}$

Exercise:

Problem: A y-intercept of $(0,3)$ and slope $\frac{2}{5}$

Solution:



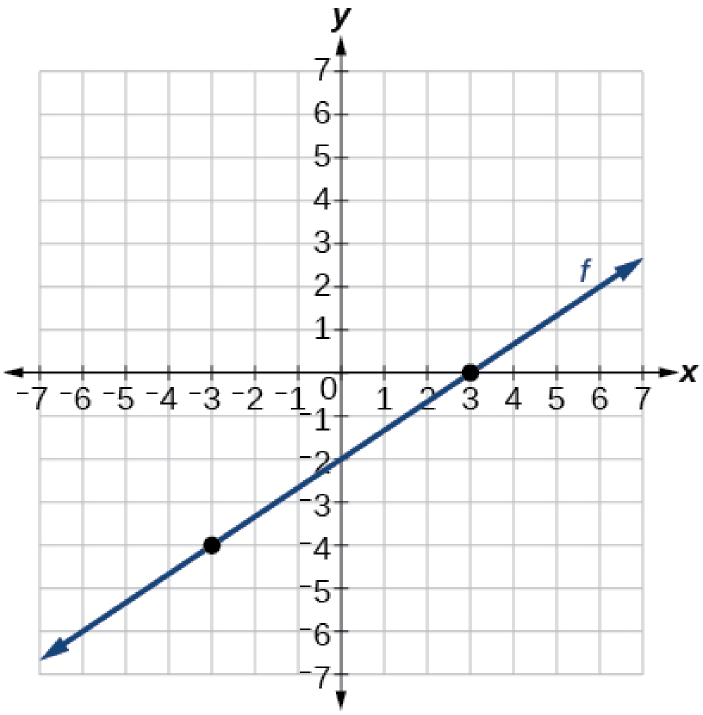
Exercise:

Problem: Passing through the points $(-6, -2)$ and $(6, -6)$

Exercise:

Problem: Passing through the points $(-3, -4)$ and $(3, 0)$

Solution:



For the following exercises, sketch the graph of each equation.

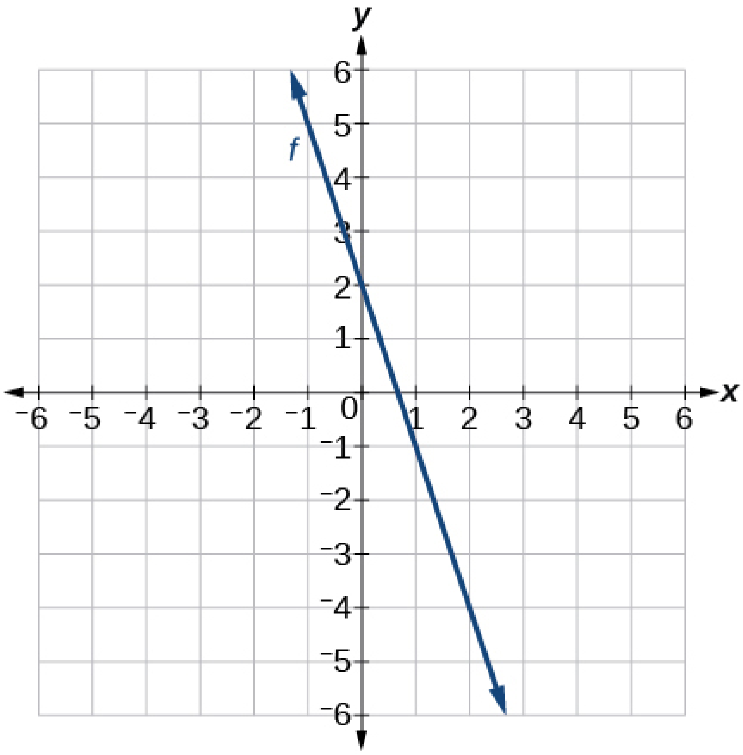
Exercise:

Problem: $f\left(x\right)=-2x-1$

Exercise:

Problem: $f\left(x\right)=-3x+2$

Solution:



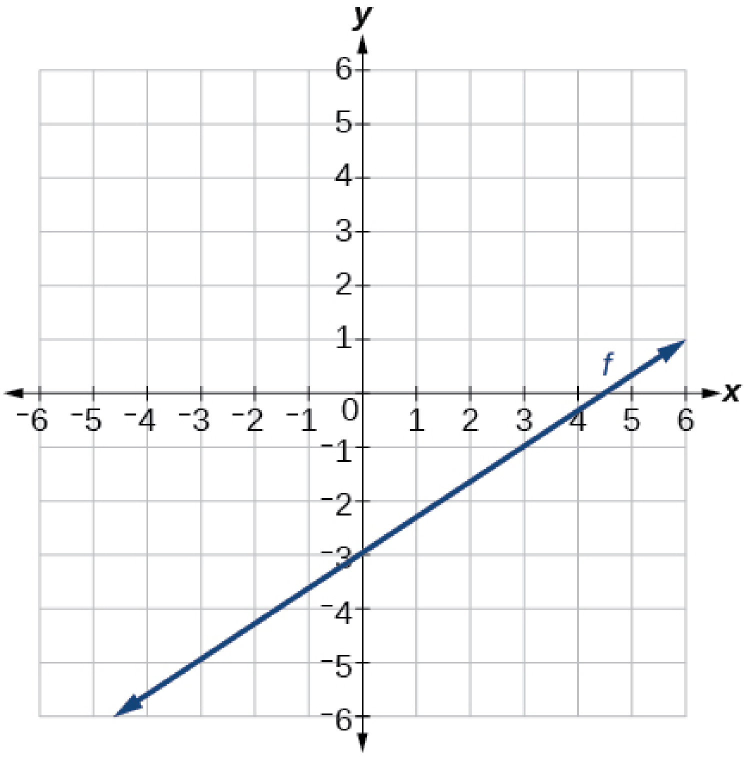
Exercise:

Problem: $f(x) = \frac{1}{3}x + 2$

Exercise:

Problem: $f(x) = \frac{2}{3}x - 3$

Solution:



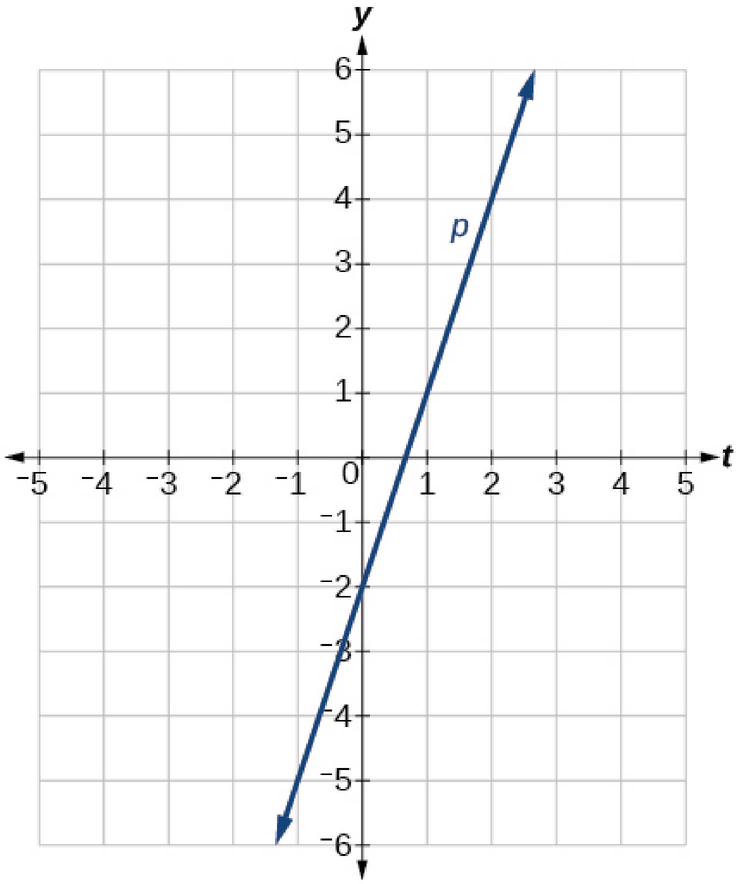
Exercise:

Problem: $f\left(t\right)=3+2t$

Exercise:

Problem: $p\left(t\right)=-2+3t$

Solution:



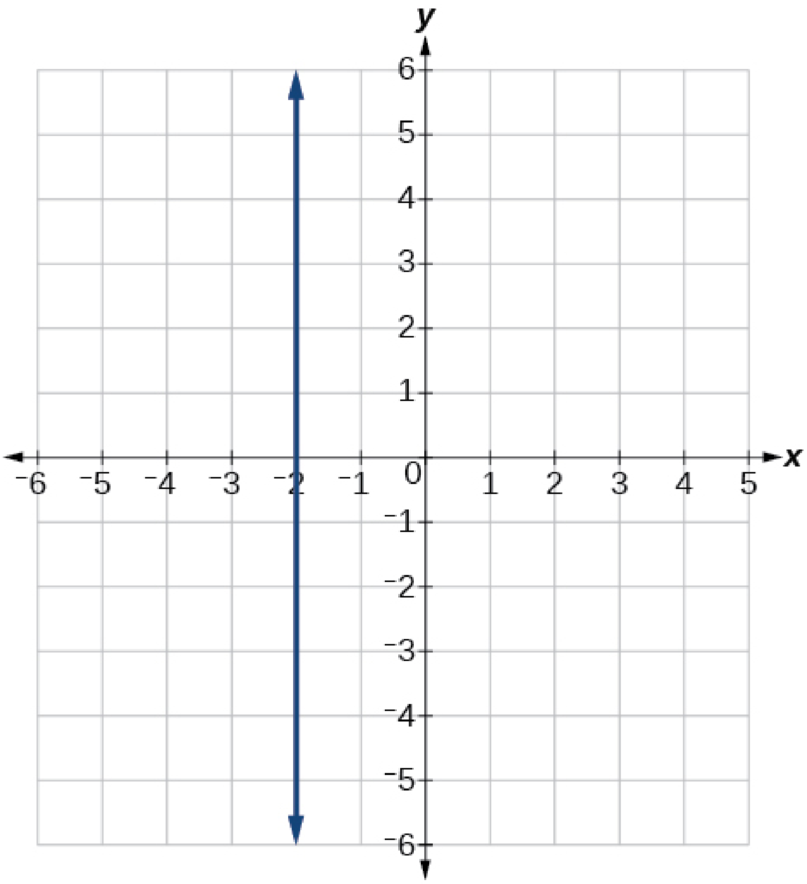
Exercise:

Problem:$x = 3$

Exercise:

Problem:$x = -2$

Solution:



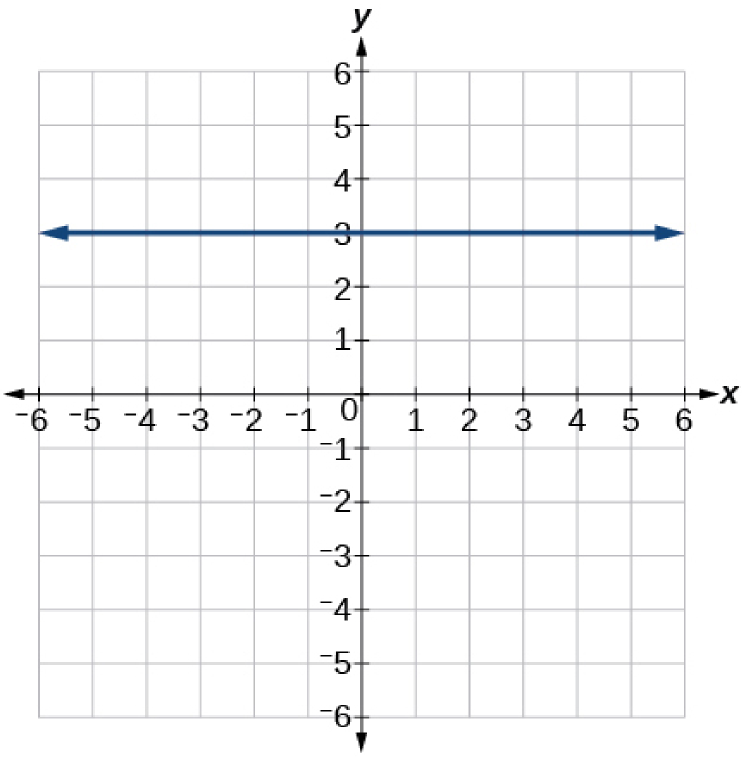
Exercise:

Problem: $r(x) = 4$

For the following exercises, write the equation of the line shown in the graph.

Exercise:

Problem:

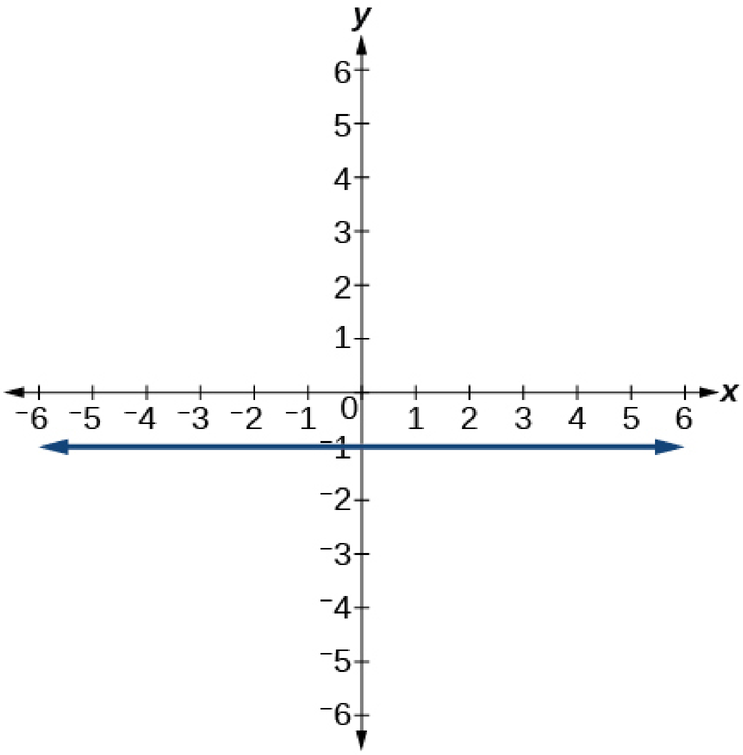


Solution:

$y = 3$

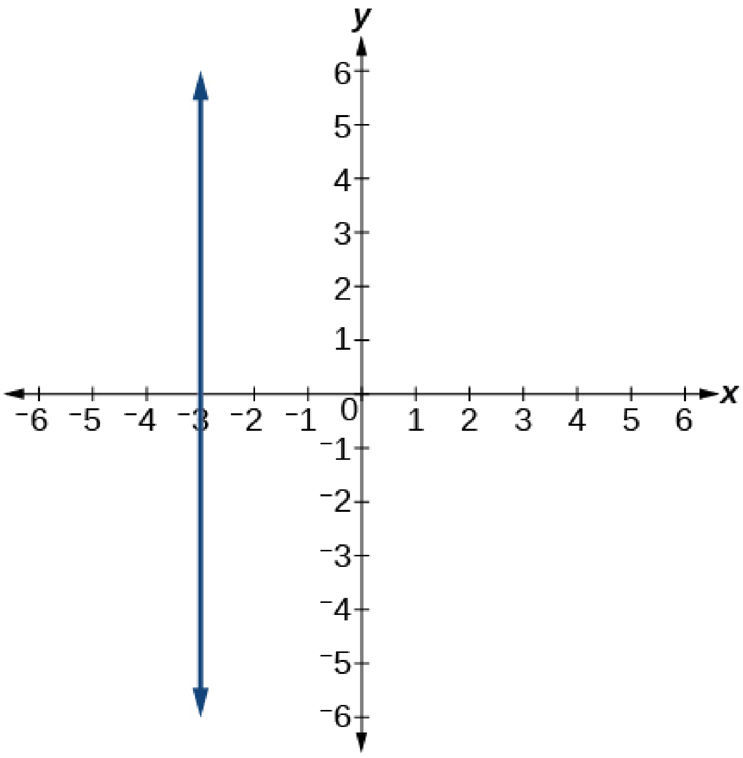
Exercise:

Problem:



Exercise:

Problem:

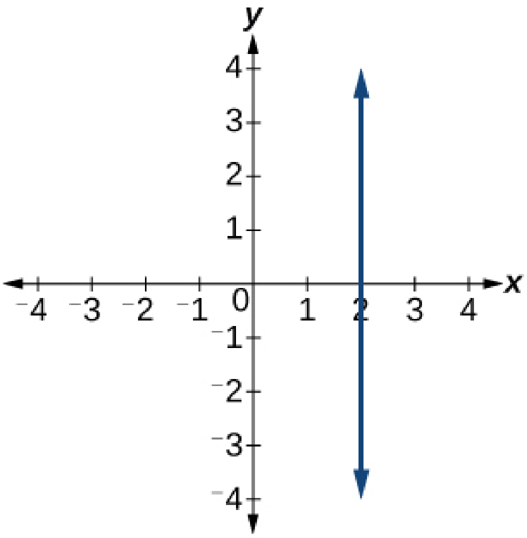


Solution:

$x = -3$

Exercise:

Problem:



Numeric

For the following exercises, which of the tables could represent a linear function? For each that could be linear, find a linear equation that models the data.

Exercise:

Problem:

x	0	5	10	15
$g(x)$	5	–10	–25	–40

Solution:

Linear, $g(x) = -3x + 5$

Exercise:

Problem:

x	0	5	10	15
$h\left(x\right)$	5	30	105	230

Exercise:

Problem:

x	0	5	10	15
$f\left(x\right)$	–5	20	45	70

Solution:

Linear, $f(x) = 5x - 5$

Exercise:

Problem:

x	5	10	20	25

$k\left(x\right)$	13	28	58	73

Exercise:

Problem:

x	0	2	4	6
$g\left(x\right)$	6	–19	–44	–69

Solution:

Linear, $g(x) = -\frac{25}{2}x + 6$

Exercise:

Problem:

x	2	4	8	10
$h\left(x\right)$	13	23	43	53

Exercise:

Problem:

x	2	4	6	8

$f(x)$	–4	16	36	56

Solution:

Linear, $f(x) = 10x - 24$

Exercise:

Problem:

x	0	2	6	8
$k(x)$	6	31	106	231

Technology

For the following exercises, use a calculator or graphing technology to complete the task.

Exercise:

Problem:

If f is a linear function, $f(0.1) = 11.5, \text{ and } f(0.4) = –5.9,$ find an equation for the function.

Solution:

$f(x) = -58x + 17.3$

Exercise:

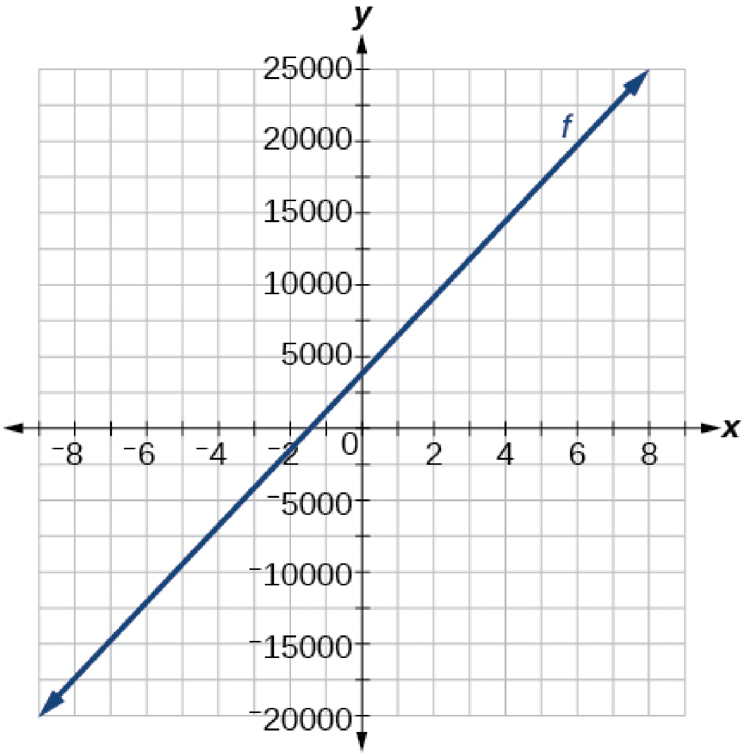
Problem:

Graph the function f on a domain of $[–10, 10] : f(x) = 0.02x - 0.01.$ Enter the function in a graphing utility. For the viewing window, set the minimum value of x to be -10 and the maximum value of x to be $10.$

Exercise:

Problem: Graph the function f on a domain of $[-10,10]:fx)=2,500x+4,000$

Solution:



Exercise:

Problem:

[link] shows the input, w,and output, k,for a linear function k. a. Fill in the missing values of the table. b. Write the linear function k,round to 3 decimal places.

w	–10	5.5	67.5	*b*
k	30	–26	*a*	–44

Solution:

$y = 3.613x - 6.129$

Exercise:

Problem:

[link] shows the input, p,and output, q,for a linear function q. a. Fill in the missing values of the table. b. Write the linear function k.

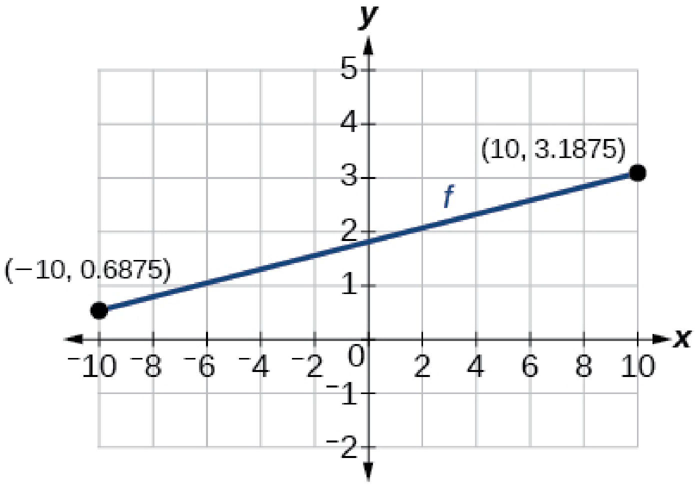
p	0.5	0.8	12	*b*
q	400	700	*a*	1,000,000

Exercise:

Problem:

Graph the linear function *f* on a domain of $[-10, 10]$ for the function whose slope is $\frac{1}{8}$ and *y*-intercept is $\frac{31}{16}$. Label the points for the input values of -10 and 10.

Solution:



Exercise:

Problem:

Graph the linear function f on a domain of $[-0.1, 0.1]$ for the function whose slope is 75 and y-intercept is -22.5. Label the points for the input values of -0.1 and 0.1.

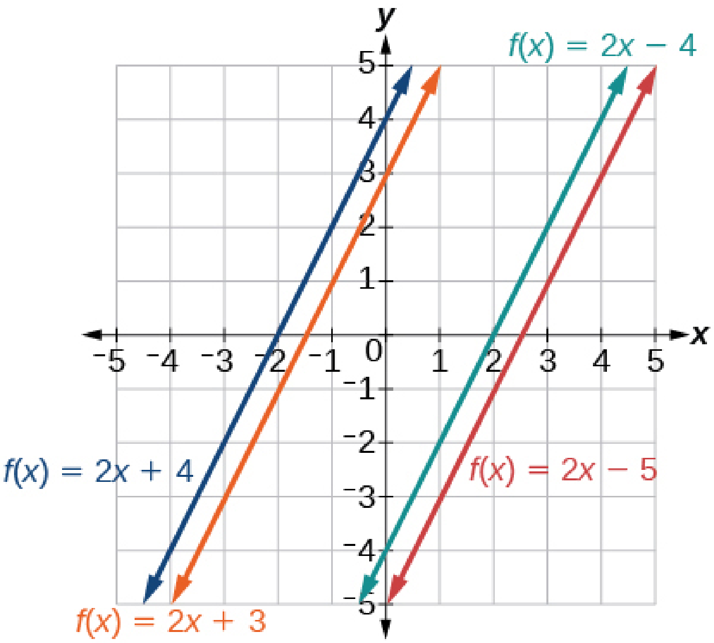
Exercise:

Problem:

Graph the linear function f where $f(x) = ax + b$ on the same set of axes on a domain of $[-4, 4]$ for the following values of a and b.

i. $a = 2; b = 3$
ii. $a = 2; b = 4$
iii. $a = 2; b = -4$
iv. $a = 2; b = -5$

Solution:



Extensions

Exercise:

Problem:

Find the value of x if a linear function goes through the following points and has the following slope: $(x, 2), (-4, 6),\ m = 3$

Exercise:

Problem:

Find the value of y if a linear function goes through the following points and has the following slope: $(10, y), (25, 100),\ m = -5$

Solution:

y = 175

Exercise:

Problem: Find the equation of the line that passes through the following points:

$(a,\ b)$ and $(a,\ b+1)$

Exercise:

Problem: Find the equation of the line that passes through the following points:

$(2a, b)$ and $(a, b+1)$

Solution:

$y = -\frac{1}{2}x + b + 2$

Exercise:

Problem: Find the equation of the line that passes through the following points:

$(a, 0)$and (c, d)

Exercise:

Problem:

Find the equation of the line parallel to the line $g(x) = -0.01x+2.01$ through the point $(1, 2)$.

Solution:

$y = –0.01x + 2.01$

Exercise:

Problem:

Find the equation of the line perpendicular to the line $g(x) = -0.01x+2.01$ through the point $(1, 2)$.

For the following exercises, use the functions $f(x) = -0.1x+200$ and $g(x) = 20x + 0.1$.

Exercise:

Problem: Find the point of intersection of the lines f and g.

Solution:

$\left(\frac{1999}{201}, \frac{400{,}001}{2010}\right)$

Exercise:

Problem: Where is $f(x)$ greater than $g(x)$? Where is $g(x)$ greater than $f(x)$?

Real-World Applications

Exercise:

Problem:

At noon, a barista notices that she has $20 in her tip jar. If she makes an average of $0.50 from each customer, how much will she have in her tip jar if she serves n more customers during her shift?

Solution:

$20 + 0.5n$

Exercise:

Problem:

A gym membership with two personal training sessions costs $125, while gym membership with five personal training sessions costs $260. What is cost per session?

Exercise:

Problem:

A clothing business finds there is a linear relationship between the number of shirts, n,it can sell and the price, p,it can charge per shirt. In particular, historical data shows that 1,000 shirts can be sold at a price of $\$30$,while 3,000 shirts can be sold at a price of $22. Find a linear equation in the form $p(n) = mn + b$ that gives the price p they can charge for n shirts.

Solution:

$p(n) = -0.004n + 34$

Exercise:

Problem:

A phone company charges for service according to the formula: $C(n) = 24 + 0.1n$,where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

Exercise:

Problem:

A farmer finds there is a linear relationship between the number of bean stalks, n, she plants and the yield, y,each plant produces. When she plants 30 stalks, each plant yields 30 oz of beans. When she plants 34 stalks, each plant produces 28 oz of beans. Find a linear relationships in the form $y = mn + b$ that gives the yield when n stalks are planted.

Solution:

$y = -0.5n + 45$

Exercise:

Problem:

A city’s population in the year 1960 was 287,500. In 1989 the population was 275,900. Compute the rate of growth of the population and make a statement about the population rate of change in people per year.

Exercise:

Problem:

A town’s population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1,700 people each year. Write an equation, $P(t)$,for the population t years after 2003.

Solution:

$P(t) = 1700t + 45,000$

Exercise:

Problem:

Suppose that average annual income (in dollars) for the years 1990 through 1999 is given by the linear function: $I(x) = 1054x + 23,286$,where x is the number of years after 1990. Which of the following interprets the slope in the context of the problem?

a. As of 1990, average annual income was $23,286.
b. In the ten-year period from 1990–1999, average annual income increased by a total of $1,054.
c. Each year in the decade of the 1990s, average annual income increased by $1,054.
d. Average annual income rose to a level of $23,286 by the end of 1999.

Exercise:

Problem:

When temperature is 0 degrees Celsius, the Fahrenheit temperature is 32. When the Celsius temperature is 100, the corresponding Fahrenheit temperature is 212. Express the Fahrenheit temperature as a linear function of C,the Celsius temperature, $F\left(C\right)$.

a. Find the rate of change of Fahrenheit temperature for each unit change temperature of Celsius.
b. Find and interpret $F(28)$.

c. Find and interpret $F(-40)$.

Solution:

a.
Rate of change $= \frac{\Delta F}{\Delta C} = \frac{212-32}{100-0} = 1.8 \text{ degrees F for one degree change in C}$

b. $F(28) = 1.8(28) + 32 = 82.4 \text{ degrees F is 28 degrees C}$

c. $F(-40) = 1.8(-40) + 32 = -40 \text{ degrees F is -40 degrees C}$

Glossary

decreasing linear function
: a function with a negative slope: If $f(x) = mx + b, \text{ then } m < 0.$

horizontal line
: a line defined by $f(x) = b$,where b is a real number. The slope of a horizontal line is 0.

increasing linear function
: a function with a positive slope: If $f(x) = mx + b, \text{ then } m > 0.$

linear function
: a function with a constant rate of change that is a polynomial of degree 1, and whose graph is a straight line

parallel lines
: two or more lines with the same slope

perpendicular lines
: two lines that intersect at right angles and have slopes that are negative reciprocals of each other

point-slope form
: the equation for a line that represents a linear function of the form $y - y_1 = m\left(x - x_1\right)$

slope
: the ratio of the change in output values to the change in input values; a measure of the steepness of a line

slope-intercept form

the equation for a line that represents a linear function in the form $f(x) = mx + b$

vertical line

a line defined by $x = a$, where a is a real number. The slope of a vertical line is undefined.

Solve Simple Interest Applications

By the end of this section, you will be able to:

- Use the simple interest formula
- Solve simple interest applications

Note:

Before you get started, take this readiness quiz.

1. Solve $0.6y=45$.
   If you missed this problem, review [link].
2. Solve $\frac{n}{1.45}=4.6$.
   If you missed this problem, review [link].

Use the Simple Interest Formula

Do you know that banks pay you to let them keep your money? The money you put in the bank is called the **principal**, P, and the bank pays you **interest**, I. The interest is computed as a certain percent of the principal; called the **rate of interest**, r. The rate of interest is usually expressed as a percent per year, and is calculated by using the decimal equivalent of the percent. The variable for time, t, represents the number of years the money is left in the account.

Note:

Simple Interest

If an amount of money, P, the principal, is invested for a period of t years at an annual interest rate r, the amount of interest, I, earned is

Equation:

$$I=Prt$$

where

Equation:

$$\begin{aligned} I &= \text{interest} \\ P &= \text{principal} \\ r &= \text{rate} \\ t &= \text{time} \end{aligned}$$

Interest earned according to this formula is called **simple interest**.

The formula we use to calculate simple interest is $I=Prt$. To use the simple interest formula we substitute in the values for variables that are given, and then solve for the unknown variable. It may be helpful to organize the information by listing all four variables and filling in the given information.

Example:

Exercise:

Problem:

Find the simple interest earned after 3 years on $\$500$ at an interest rate of 6%.

Solution:

Solution

Organize the given information in a list.

$$\begin{aligned} I &= ? \\ P &= \$500 \\ r &= 6\% \\ t &= 3 \text{ years} \end{aligned}$$

We will use the simple interest formula to find the interest.

Write the formula.	$I = Prt$
Substitute the given information. Remember to write the percent in decimal form.	$I = (500)(0.06)(3)$
Simplify.	$I = 90$
Check your answer. Is \$90 a reasonable interest earned on \$500 in 3 years?	
In 3 years the money earned 18%. If we rounded to 20%, the interest would have been 500(0.20) or \$100. Yes, \$90 is reasonable.	
Write a complete sentence that answers the question.	The simple interest is \$90.

Note:
Exercise:

Problem:

Find the simple interest earned after 4 years on $\$800$ at an interest rate of 5%.

Solution:

$160

Note:
Exercise:

Problem:

Find the simple interest earned after 2 years on $\$700$ at an interest rate of 4%.

Solution:

$56

In the next example, we will use the simple interest formula to find the principal.

Example:
Exercise:

Problem:

Find the principal invested if $\$178$ interest was earned in 2 years at an interest rate of 4%.

Solution:
Solution

Organize the given information in a list.

$$\begin{aligned} I &= \$178 \\ P &= ? \\ r &= 4\% \\ t &= 2 \text{ years} \end{aligned}$$

We will use the simple interest formula to find the principal.

Write the formula.	$I = Prt$
Substitute the given information.	$178 = P(0.04)(2)$
Divide.	$\frac{178}{0.08} = \frac{0.08P}{0.08}$
Simplify.	$2{,}225 = P$
Check your answer. Is it reasonable that \$2,225 would earn \$178 in 2 years?	
$I = Prt$	
$178 \overset{?}{=} 2{,}225(0.04)(2)$	
$178 = 178 \checkmark$	
Write a complete sentence that answers the question.	The principal is \$2,225.

Note:
Exercise:

Problem:

Find the principal invested if $495 interest was earned in 3 years at an interest rate of 6%.

Solution:

$2,750

Note:
Exercise:

Problem:

Find the principal invested if $1,246 interest was earned in 5 years at an interest rate of 7%.

Solution:

$3,560

Now we will solve for the rate of interest.

Example:
Exercise:

Problem:

Find the rate if a principal of $8,200 earned $3,772 interest in 4 years.

Solution:
Solution

Organize the given information.

$$\begin{aligned} I &= \$3{,}772 \\ P &= \$8{,}200 \\ r &= ? \\ t &= 4 \text{ years} \end{aligned}$$

We will use the simple interest formula to find the rate.

Write the formula.	$I = Prt$
Substitute the given information.	$3{,}772 = 8{,}200r(4)$
Multiply.	$3{,}772 = 32{,}800r$
Divide.	$\frac{3{,}772}{32{,}800} = \frac{32{,}800r}{32{,}800}$
Simplify.	$0.115 = r$
Write as a percent.	$11.5\% = r$
Check your answer. Is 11.5% a reasonable rate if \$3,772 was earned in 4 years?	
$I = Prt$	

$3{,}772 \overset{?}{=} 8{,}200(0.115)(4)$	
$3{,}772 = 3{,}772\checkmark$	
Write a complete sentence that answers the question.	The rate was 11.5%.

Note:
Exercise:

Problem:

Find the rate if a principal of \$5,000 earned \$1,350 interest in 6 years.

Solution:

4.5%

Note:
Exercise:

Problem:

Find the rate if a principal of \$9,000 earned \$1,755 interest in 3 years.

Solution:

6.5%

Solve Simple Interest Applications

Applications with simple interest usually involve either investing money or borrowing money. To solve these applications, we continue to use the same strategy for applications that we have used earlier in this chapter. The only difference is that in place of translating to get an equation, we can use the simple interest formula.

We will start by solving a simple interest application to find the interest.

Example:
Exercise:

Problem:

Nathaly deposited $\$12{,}500$ in her bank account where it will earn 4% interest. How much interest will Nathaly earn in 5 years?

Solution:
Solution

We are asked to find the Interest, I.

Organize the given information in a list.

$$\begin{aligned} I &= ? \\ P &= \$12{,}500 \\ r &= 4\% \\ t &= 5 \text{ years} \end{aligned}$$

Write the formula.	$I = Prt$

Substitute the given information.	$I = (12{,}500)(0.04)(5)$
Simplify.	$I = 2{,}500$
Check your answer. Is $2,500 a reasonable interest on $12,500 over 5 years?	
At 4% interest per year, in 5 years the interest would be 20% of the principal. Is 20% of $12,500 equal to $2,500? Yes.	
Write a complete sentence that answers the question.	The interest is $2,500.

Note:
Exercise:

Problem:

Areli invested a principal of $\$950$ in her bank account with interest rate 3%. How much interest did she earn in 5 years?

Solution:

$142.50

Note:
Exercise:

Problem:

Susana invested a principal of $\$36,000$ in her bank account with interest rate 6.5%. How much interest did she earn in 3 years?

Solution:

$7,020

There may be times when you know the amount of interest earned on a given principal over a certain length of time, but you don't know the rate. For instance, this might happen when family members lend or borrow money among themselves instead of dealing with a bank. In the next example, we'll show how to solve for the rate.

Example:
Exercise:

Problem:

Loren lent his brother $\$3,000$ to help him buy a car. In 4 years his brother paid him back the $\$3,000$ plus $\$660$ in interest. What was the rate of interest?

Solution:
Solution

We are asked to find the rate of interest, r.

Organize the given information.

$$\begin{aligned} I &= 660 \\ P &= \$3{,}000 \\ r &= ? \\ t &= 4 \text{ years} \end{aligned}$$

Write the formula.	$I = Prt$
Substitute the given information.	$660 = (3{,}000)r(4)$
Multiply.	$660 = (12{,}000)r$
Divide.	$\frac{660}{12{,}000} = \frac{(12{,}000)r}{12{,}000}$
Simplify.	$0.055 = r$
Change to percent form.	$5.5\% = r$
Check your answer. Is 5.5% a reasonable interest rate to pay your brother?	
$I = Prt$	
$660 \overset{?}{=} (3{,}000)(0.055)(4)$	
$660 = 660\checkmark$	
Write a complete sentence that answers the question.	The rate of interest was 5.5%.

Note:
Exercise:

Problem:

Jim lent his sister $5,000 to help her buy a house. In 3 years, she paid him the $5,000, plus $900 interest. What was the rate of interest?

Solution:

6%

Note:
Exercise:

Problem:

Hang borrowed $7,500 from her parents to pay her tuition. In 5 years, she paid them $1,500 interest in addition to the $7,500 she borrowed. What was the rate of interest?

Solution:

4%

There may be times when you take a loan for a large purchase and the amount of the principal is not clear. This might happen, for instance, in making a car purchase when the dealer adds the cost of a warranty to the price of the car. In the next example, we will solve a simple interest application for the principal.

Example:
Exercise:

Problem:

Eduardo noticed that his new car loan papers stated that with an interest rate of 7.5%, he would pay $\$6{,}596.25$ in interest over 5 years. How much did he borrow to pay for his car?

Solution:
Solution

We are asked to find the principal, P.

Organize the given information.

$$\begin{array}{rcl} I & = & 6{,}596.25 \\ P & = & ? \\ r & = & 7.5\% \\ t & = & 5 \text{ years} \end{array}$$

Write the formula.	$I = Prt$
Substitute the given information.	$6{,}596.25 = P(0.075)(5)$
Multiply.	$6{,}596.25 = 0.375P$
Divide.	$\frac{6{,}596.25}{0.375} = \frac{0.375P}{0.375}$
Simplify.	$17{,}590 = P$
Check your answer. Is \$17,590 a	

reasonable amount to borrow to buy a car?	
$I = Prt$	
$6{,}596.25 \overset{?}{=} (17{,}590)(0.075)(5)$	
$6{,}596.25 = 6{,}596.25 \checkmark$	
Write a complete sentence that answers the question.	The amount borrowed was \$17,590.

Note:
Exercise:

Problem:

Sean's new car loan statement said he would pay $\$4{,}866.25$ in interest from an interest rate of 8.5% over 5 years. How much did he borrow to buy his new car?

Solution:

\$11,450

Note:
Exercise:

Problem:

In 5 years, Gloria's bank account earned $\$2{,}400$ interest at 5%. How much had she deposited in the account?

Solution:

$9,600

In the simple interest formula, the rate of interest is given as an annual rate, the rate for one year. So the units of time must be in years. If the time is given in months, we convert it to years.

Example:
Exercise:

Problem:

Caroline got $\$900$ as graduation gifts and invested it in a 10-month certificate of deposit that earned 2.1% interest. How much interest did this investment earn?

Solution:
Solution

We are asked to find the interest, I.

Organize the given information.

$$\begin{array}{rcl} I & = & ? \\ P & = & \$900 \\ r & = & 2.1\% \\ t & = & 10 \text{ months} \end{array}$$

Write the formula.	$I = Prt$
Substitute the given information, converting 10 months to $\frac{10}{12}$ of a year.	$I = \$900(0.021)\left(\frac{10}{12}\right)$
Multiply.	$I = 15.75$
Check your answer. Is \$15.75 a reasonable amount of interest?	
If Caroline had invested the \$900 for a full year at 2% interest, the amount of interest would have been \$18. Yes, \$15.75 is reasonable.	
Write a complete sentence that answers the question.	The interest earned was \$15.75.

Note:
Exercise:

Problem:

Adriana invested \$4,500 for 8 months in an account that paid 1.9% interest. How much interest did she earn?

Solution:

\$57.00

Note:
Exercise:

Problem:

Milton invested $\$2,460$ for 20 months in an account that paid 3.5% interest How much interest did he earn?

Solution:

$143.50

Key Concepts

- **Simple interest**
  - If an amount of money, P, the principal, is invested for a period of t years at an annual interest rate r, the amount of interest, I, earned is $I = Prt$
  - Interest earned according to this formula is called **simple interest**.

Practice Makes Perfect

Use the Simple Interest Formula

In the following exercises, use the simple interest formula to fill in the missing information.

Exercise:

Problem:

Interest	Principal	Rate	Time (years)
	$1200	3%	5

Solution:

$180

Exercise:

Problem:

Interest	Principal	Rate	Time (years)
	$1500	2%	4

Exercise:

Problem:

Interest	Principal	Rate	Time (years)
$4410		4.5%	7

Solution:

$14,000

Exercise:

Problem:

Interest	Principal	Rate	Time (years)
$2212		3.2%	6

Exercise:

Problem:

Interest	Principal	Rate	Time (years)
$577.08	$4580		2

Solution:

6.3%

Exercise:

Problem:

Interest	Principal	Rate	Time (years)
$\$528.12$	$\$3260$		3

In the following exercises, solve the problem using the simple interest formula.

Exercise:

Problem:

Find the simple interest earned after 5 years on $\$600$ at an interest rate of 3%.

Solution:

$90

Exercise:

Problem:

Find the simple interest earned after 4 years on $\$900$ at an interest rate of 6%.

Exercise:

Problem:

Find the simple interest earned after 2 years on $\$8,950$ at an interest rate of 3.24%.

Solution:

$579.96

Exercise:

Problem:

Find the simple interest earned after 3 years on $\$6,510$ at an interest rate of 2.85%.

Exercise:

Problem:

Find the simple interest earned after 8 years on $\$15,500$ at an interest rate of 11.425%.

Solution:

$14,167

Exercise:

Problem:

Find the simple interest earned after 6 years on $\$23,900$ at an interest rate of 12.175%.

Exercise:

Problem:

Find the principal invested if $\$656$ interest was earned in 5 years at an interest rate of 4%.

Solution:

$3,280

Exercise:

Problem:

Find the principal invested if $\$177$ interest was earned in 2 years at an interest rate of 3%.

Exercise:

Problem:

Find the principal invested if $\$70.95$ interest was earned in 3 years at an interest rate of 2.75%.

Solution:

$860

Exercise:

Problem:

Find the principal invested if $\$636.84$ interest was earned in 6 years at an interest rate of 4.35%.

Exercise:

Problem:

Find the principal invested if $\$15,222.57$ interest was earned in 6 years at an interest rate of 10.28%.

Solution:

$24,679.91

Exercise:

Problem:

Find the principal invested if $\$10,953.70$ interest was earned in 5 years at an interest rate of 11.04%.

Exercise:

Problem:

Find the rate if a principal of $\$5,400$ earned $\$432$ interest in 2 years.

Solution:

4%

Exercise:

Problem:

Find the rate if a principal of $\$2,600$ earned $\$468$ interest in 6 years.

Exercise:

Problem:

Find the rate if a principal of $\$11,000$ earned $\$1,815$ interest in 3 years.

Solution:

5.5%

Exercise:

Problem:

Find the rate if a principal of $\$8,500$ earned $\$3,230$ interest in 4 years.

Solve Simple Interest Applications

In the following exercises, solve the problem using the simple interest formula.

Exercise:

Problem:

Casey deposited $\$1,450$ in a bank account with interest rate 4%. How much interest was earned in 2 years?

Solution:

$116

Exercise:

Problem:

Terrence deposited $\$5,720$ in a bank account with interest rate 6%. How much interest was earned in 4 years?

Exercise:

Problem:

Robin deposited $\$31,000$ in a bank account with interest rate 5.2%. How much interest was earned in 3 years?

Solution:

$4,836

Exercise:

Problem:

Carleen deposited $\$16,400$ in a bank account with interest rate 3.9%. How much interest was earned in 8 years?

Exercise:

Problem:

Hilaria borrowed $\$8,000$ from her grandfather to pay for college. Five years later, she paid him back the $\$8,000$, plus $\$1,200$ interest. What was the rate of interest?

Solution:

3%

Exercise:

Problem:

Kenneth lent his niece $\$1,200$ to buy a computer. Two years later, she paid him back the $\$1,200$, plus $\$96$ interest. What was the rate of interest?

Exercise:

Problem:

Lebron lent his daughter $\$20,000$ to help her buy a condominium. When she sold the condominium four years later, she paid him the $\$20,000$, plus $\$3,000$ interest. What was the rate of interest?

Solution:

3.75%

Exercise:

Problem:

Pablo borrowed $\$50,000$ to start a business. Three years later, he repaid the $\$50,000$, plus $\$9,375$ interest. What was the rate of interest?

Exercise:

Problem:

In 10 years, a bank account that paid 5.25% earned $\$18,375$ interest. What was the principal of the account?

Solution:

$35,000

Exercise:

Problem:

In 25 years, a bond that paid 4.75% earned $\$2,375$ interest. What was the principal of the bond?

Exercise:

Problem:

Joshua's computer loan statement said he would pay $\$1,244.34$ in interest for a 3 year loan at 12.4%. How much did Joshua borrow to buy the computer?

Solution:

$3,345

Exercise:

Problem:

Margaret's car loan statement said she would pay $\$7,683.20$ in interest for a 5 year loan at 9.8%. How much did Margaret borrow to buy the car?

Exercise:

Problem:

Caitlin invested $\$8,200$ in an 18-month certificate of deposit paying 2.7% interest. How much interest did she earn form this investment?

Solution:

$332.10

Exercise:

Problem:

Diego invested $\$6,100$ in a 9-month certificate of deposit paying 1.8% interest. How much interest did he earn form this investment?

Exercise:

Problem:

Airin borrowed $\$3,900$ from her parents for the down payment on a car and promised to pay them back in 15 months at a 4% rate of interest. How much interest did she owe her parents?

Solution:

$195.00

Exercise:

Problem:

Yuta borrowed $\$840$ from his brother to pay for his textbooks and promised to pay him back in 5 months at a 6% rate of interest. How much interest did Yuta owe his brother?

Everyday Math

Exercise:

Problem:

Interest on savings Find the interest rate your local bank pays on savings accounts.

ⓐ What is the interest rate?
ⓑ Calculate the amount of interest you would earn on a principal of $\$8,000$ for 5 years.

Solution:

Answers will vary.

Exercise:

Problem:

Interest on a loan Find the interest rate your local bank charges for a car loan.

ⓐ What is the interest rate?
ⓑ Calculate the amount of interest you would pay on a loan of \$8,000 for 5 years.

Writing Exercises

Exercise:

Problem:

Why do banks pay interest on money deposited in savings accounts?

Solution:

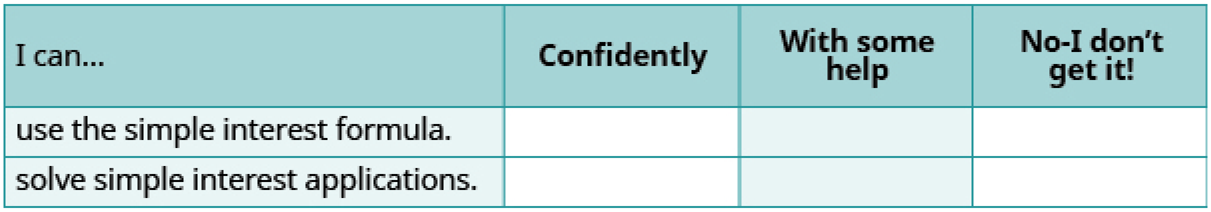
Answers will vary.

Exercise:

Problem: Why do banks charge interest for lending money?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don’t get it!
use the simple interest formula.			
solve simple interest applications.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

simple interest

If an amount of money, P, the principal, is invested for a period of t years at an annual interest rate r, the amount of interest, I, earned is $I = Prt$. Interest earned according to this formula is called simple interest.

Use a General Strategy to Solve Linear Equations
By the end of this section, you will be able to:

- Solve linear equations using a general strategy
- Classify equations
- Solve equations with fraction or decimal coefficients

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\frac{3}{2}\left(12x+20\right)$.
   If you missed this problem, review [link].
2. Simplify: $5-2\left(n+1\right)$.
   If you missed this problem, review [link].
3. Find the LCD of $\frac{5}{6}$ and $\frac{1}{4}$.
   If you missed this problem, review [link].

Solve Linear Equations Using a General Strategy

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that makes it a true statement. Any value of the variable that makes the equation true is called a **solution** to the equation. It is the answer to the puzzle!

Note:
Solution of an Equation
A **solution** of an equation is a value of a variable that makes a true statement when substituted into the equation.

To determine whether a number is a solution to an equation, we substitute the value for the variable in the equation. If the resulting equation is a true statement, then the number is a solution of the equation.

Note:
Determine Whether a Number is a Solution to an Equation.

Substitute the number for the variable in the equation.
Simplify the expressions on both sides of the equation.
Determine whether the resulting equation is true.

- If it is true, the number is a solution.
- If it is not true, the number is not a solution.

Example:
Exercise:

Problem:

Determine whether the values are solutions to the equation: $5y+3=10y-4$.

ⓐ $y=\frac{3}{5}$ ⓑ $y=\frac{7}{5}$

Solution:

Since a solution to an equation is a value of the variable that makes the equation true, begin by substituting the value of the solution for the variable.

ⓐ

	$5y+3=10y-4$
Substitute $\frac{3}{5}$ for y.	$5\left(\frac{3}{5}\right)+3\overset{?}{=}10\left(\frac{3}{5}\right)-4$
Multiply.	$3+3\overset{?}{=}6-4$
Simplify.	$6\neq 2$

Since $y=\frac{3}{5}$ does not result in a true equation, $y=\frac{3}{5}$ is not a solution to the equation $5y+3=10y-4$.

ⓑ

	$5y+3=10y-4$
Substitute $\frac{7}{5}$ for y.	$5\left(\frac{7}{5}\right)+3\overset{?}{=}10\left(\frac{7}{5}\right)-4$
Multiply.	$7+3\overset{?}{=}14-4$

Simplify.	$10 = 10$ ✓

Since $y = \frac{7}{5}$ results in a true equation, $y = \frac{7}{5}$ is a solution to the equation $5y + 3 = 10y - 4$.

Note:
Exercise:

Problem:

Determine whether the values are solutions to the equation: $9y + 2 = 6y + 3$.

ⓐ $y = \frac{4}{3}$ ⓑ $y = \frac{1}{3}$

Solution:

ⓐ no ⓑ yes

Note:
Exercise:

Problem:

Determine whether the values are solutions to the equation: $4x - 2 = 2x + 1$.

ⓐ $x = \frac{3}{2}$ ⓑ $x = -\frac{1}{2}$

Solution:

ⓐ yes ⓑ no

There are many types of equations that we will learn to solve. In this section we will focus on a **linear equation**.

Note:
Linear Equation
A **linear equation** is an equation in one variable that can be written, where a and b are real numbers and $a \neq 0$, as:
Equation:

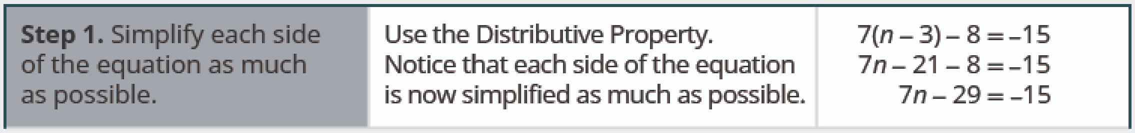
$$ax + b = 0$$

To solve a linear equation it is a good idea to have an overall strategy that can be used to solve any linear equation. In the next example, we will give the steps of a general strategy for solving any linear equation. Simplifying each side of the equation as much as possible first makes the rest of the steps easier.

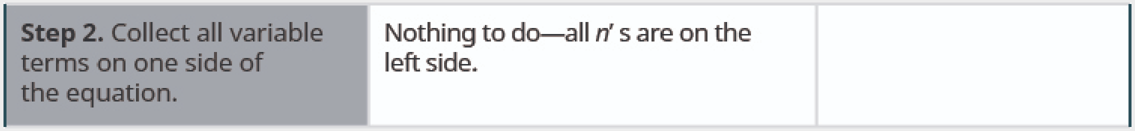
Example:
How to Solve a Linear Equation Using a General Strategy
Exercise:

Problem: Solve: $7\left(n-3\right)-8=-15$.

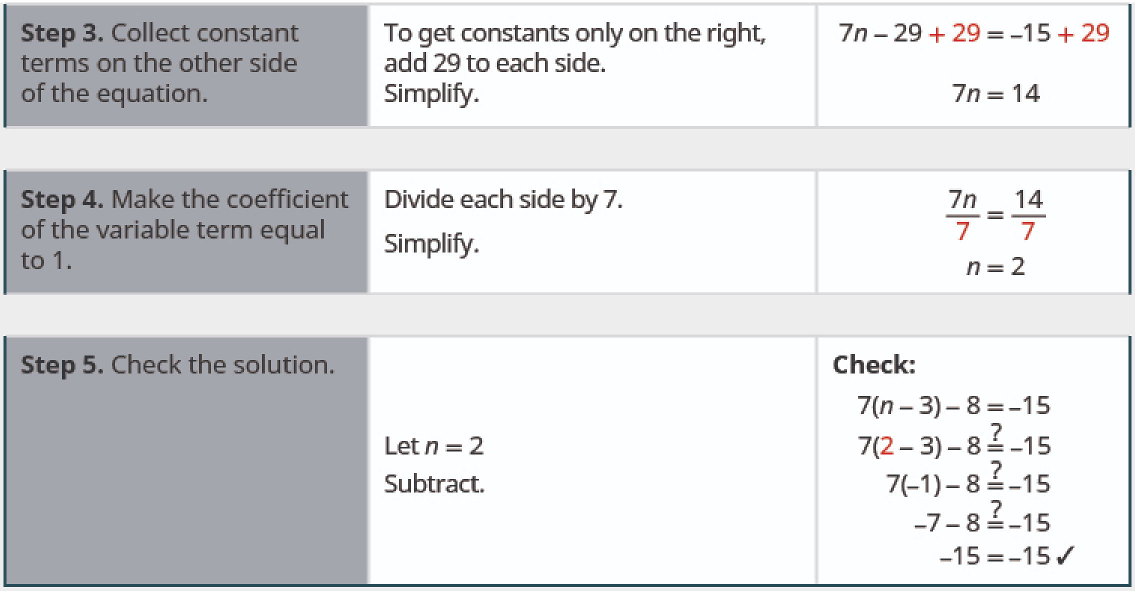
Solution:



Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is now simplified as much as possible.	$7(n-3)-8=-15$ $7n-21-8=-15$ $7n-29=-15$



Step 2. Collect all variable terms on one side of the equation.	Nothing to do—all n' s are on the left side.	



Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 29 to each side. Simplify.	$7n - 29 + 29 = -15 + 29$ $7n = 14$
Step 4. Make the coefficient of the variable term equal to 1.	Divide each side by 7. Simplify.	$\frac{7n}{7} = \frac{14}{7}$ $n = 2$
Step 5. Check the solution.	Let $n = 2$ Subtract.	**Check:** $7(n - 3) - 8 = -15$ $7(2 - 3) - 8 \overset{?}{=} -15$ $7(-1) - 8 \overset{?}{=} -15$ $-7 - 8 \overset{?}{=} -15$ $-15 = -15$ ✓

Note:
Exercise:

Problem: Solve: $2\left(m - 4\right) + 3 = -1.$

Solution:

$m = 2$

Note:
Exercise:

Problem: Solve: $5\left(a - 3\right) + 5 = -10.$

Solution:

$a = 0$

These steps are summarized in the General Strategy for Solving Linear Equations below.

Note:
Solve linear equations using a general strategy.

Simplify each side of the equation as much as possible.	Use the Distributive Property to remove any parentheses.	Combine like terms.
Collect all the variable terms on one side of the equation.	Use the Addition or Subtraction Property of Equality.	
Collect all the constant terms on the other side of the equation.	Use the Addition or Subtraction Property of Equality.	
Make the coefficient of the variable term equal to 1.	Use the Multiplication or Division Property of Equality.	State the solution to the equation.
Check the solution.	Substitute the solution into the original equation to make sure the result is a true statement.	

Example:
Exercise:

Problem: Solve: $\frac{2}{3}(3m-6)=5-m$.

Solution:

	$\frac{2}{3}(3m-6)=5-m$

Distribute.		$2m - 4 = 5 - m$
Add *m* to both sides to get the variables only on the left.		$2m + m - 4 = 5 - m + m$
Simplify.		$3m - 4 = 5$
Add 4 to both sides to get constants only on the right.		$3m - 4 + 4 = 5 + 4$
Simplify.		$3m = 9$
Divide both sides by three.		$\frac{3m}{3} = \frac{9}{3}$
Simplify.		$m = 3$
Check:	$\frac{2}{3}(3m - 6) = 5 - m$	
Let $m = 3$.	$\frac{2}{3}(3 \cdot 3 - 6) \overset{?}{=} 5 - 3$	
	$\frac{2}{3}(9 - 6) \overset{?}{=} 2$	

	$\frac{2}{3}(3) \stackrel{?}{=} 2$	
	$2 = 2$ ✓	

Note:
Exercise:

Problem: Solve: $\frac{1}{3}(6u + 3) = 7 - u$.

Solution:

$u = 2$

Note:
Exercise:

Problem: Solve: $\frac{2}{3}(9x - 12) = 8 + 2x$.

Solution:

$x = 4$

We can solve equations by getting all the variable terms to either side of the equal sign. By collecting the variable terms on the side where the coefficient of the variable is larger, we avoid working with some negatives.

This will be a good strategy when we solve inequalities later in this chapter. It also helps us prevent errors with negatives.

Example:
Exercise:

Problem: Solve: $4\left(x-1\right)-2=5\left(2x+3\right)+6.$

Solution:

			$4(x-1)-2=5(2x+3)+6$
Distribute.			$4x-4-2=10x+15+6$
Combine like terms.			$4x-6=10x+21$
Subtract $4x$ from each side to get the variables only on the right since $10>4.$			$4x-4x-6=10x-4x+21$
Simplify.			$-6=6x+21$
Subtract 21 from each side to get the constants on left.			$-6-21=6x+21-21$
Simplify.			$-27=6x$

Divide both sides by 6.				$\frac{-27}{6}=\frac{6x}{6}$
Simplify.				$-\frac{9}{2}=x$
Check:	$4(x-1)-2=5(2x+3)+6$			
Let $x=-\frac{9}{2}$.	$4\left(-\frac{9}{2}-1\right)-2\overset{?}{=}5\left(2\left(-\frac{9}{2}\right)+3\right)+6$			
	$4\left(-\frac{11}{2}\right)-2\overset{?}{=}5(-9+3)+6$			
	$-22-2\overset{?}{=}5(-6)+6$			
	$-24\overset{?}{=}-30+6$			
	$-24=-24$ ✓			

Note:
Exercise:

Problem: Solve: $6\left(p-3\right)-7=5\left(4p+3\right)-12.$

Solution:

$p = -2$

Note:
Exercise:

Problem: Solve: $8\left(q+1\right)-5=3\left(2q-4\right)-1$.

Solution:

$q = -8$

Example:
Exercise:

Problem: Solve: $10\left[3-8\left(2s-5\right)\right]=15\left(40-5s\right)$.

Solution:

	$10[3-8(2s-5)]=15(40-5s)$
Simplify from the innermost parentheses first.	$10[3-16s+40]=15(40-5s)$

Combine like terms in the brackets.	$10[43-16s]=15(40-5s)$	
Distribute.	$430-160s=600-75s$	
Add $160s$ to both sides to get the variables to the right.	$430-160s+160s=600-75s+160s$	
Simplify.	$430=600+85s$	
Subtract 600 from both sides to get the constants to the left.	$430-600=600+85s-600$	
Simplify.	$-170=85s$	
Divide both sides by 85.	$\frac{-170}{85}=\frac{85s}{85}$	
Simplify.	$-2=s$	
Check:	$10[3-8(2s-5)]=15(40-5s)$	
Let $s=-2$.	$10[3-8(2(-2)-5)]\overset{?}{=}15(40-5(-2))$	
	$10[3-8(-4-5)]\overset{?}{=}15(40+10)$	
	$10[3-8(-9)]\overset{?}{=}15(50)$	

	$10(3+72) \stackrel{?}{=} 750$	
	$10(75) \stackrel{?}{=} 750$	
	$750 = 750$ ✓	

Note:
Exercise:

Problem: Solve: $6\left[4-2\left(7y-1\right)\right]=8\left(13-8y\right)$.

Solution:

$y=-\frac{17}{5}$

Note:
Exercise:

Problem: Solve: $12\left[1-5\left(4z-1\right)\right]=3\left(24+11z\right)$.

Solution:

$z=0$

Classify Equations

Whether or not an equation is true depends on the value of the variable. The equation $7x+8=-13$ is true when we replace the variable, *x*, with the value -3, but not true when we replace *x* with any other value. An equation like this is called a **conditional equation**. All the equations we have solved so far are conditional equations.

Note:
Conditional Equation
An equation that is true for one or more values of the variable and false for all other values of the variable is a **conditional equation**.

Now let’s consider the equation $7y+14=7\left(y+2\right)$. Do you recognize that the left side and the right side are equivalent? Let’s see what happens when we solve for *y*.

Solve:

	$7y+14=7(y+2)$
Distribute.	$7y+14=7y+14$
Subtract $7y$ to each side to get the y's to one side.	$7y-7y+14=7y-7y+14$

Simplify—the *y*’s are eliminated.	$14 = 14$
	But $14 = 14$ is true.

This means that the equation $7y + 14 = 7\left(y + 2\right)$ is true for any value of *y*. We say the solution to the equation is all of the real numbers. An equation that is true for any value of the variable is called an **identity**.

Note:
Identity
An equation that is true for any value of the variable is called an **identity**. The solution of an identity is all real numbers.

What happens when we solve the equation $-8z = -8z + 9$?

Solve:

	$-8z = -8z + 9$
Add $8z$ to both sides to leave the constant alone on the right.	$-8z + 8z = -8z + 8z + 9$
Simplify—the z′s are eliminated.	$0 \neq 9$
	But $0 \neq 9$.

Solving the equation $-8z=-8z+9$ led to the false statement $0=9$. The equation $-8z=-8z+9$ will not be true for any value of z. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a **contradiction**.

Note:
Contradiction
An equation that is false for all values of the variable is called a **contradiction**.
A contradiction has no solution.

The next few examples will ask us to classify an equation as conditional, an identity, or as a contradiction.

Example:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:
$6\left(2n-1\right)+3=2n-8+5\left(2n+1\right)$.

Solution:

	$6(2n-1)+3=2n-8+5(2n+1)$

Distribute.	$12n-6+3=2n-8+10n+5$
Combine like terms.	$12n-3=12n-3$
Subtract $12n$ from each side to get the *n*’s to one side.	$12n-12n-3=12n-12n-3$
Simplify.	$-3=-3$
This is a true statement.	The equation is an identity.
	The solution is all real numbers.

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $4+9\left(3x-7\right)=-42x-13+23\left(3x-2\right)$.

Solution:

identity; all real numbers

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:
$8\left(1-3x\right)+15\left(2x+7\right)=2\left(x+50\right)+4\left(x+3\right)+1.$

Solution:

identity; all real numbers

Example:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $8+3(a-4)=0.$

Solution:

		$8+3(a-4)=0$
Distribute.		$8+3a-12=0$
Combine like terms.		$3a-4=0$

Add 4 to both sides.		$3a-4+4=0+4$
Simplify.		$3a=4$
Divide.		$\frac{3a}{3}=\frac{4}{3}$
Simplify.		$a=\frac{4}{3}$
The equation is true when $a=\frac{4}{3}$.	This is a conditional equation.	
	The solution is $a=\frac{4}{3}$.	

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $11\left(q+3\right)-5=19$.

Solution:

conditional equation; $q=-\frac{9}{11}$

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6+14\left(k-8\right)=95$.

Solution:

conditional equation; $k=\frac{201}{14}$

Example:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $5m+3\left(9+3m\right)=2\left(7m-11\right)$.

Solution:

	$5m+3(9+3m)=2(7m-11)$
Distribute.	$5m+27+9m=14m-22$
Combine like terms.	$14m+27=14m-22$

Subtract $14m$ from both sides.	$14m + 27 - 14m = 14m - 22 - 14m$
Simplify.	$27 \neq -22$
But $27 \neq -22$.	The equation is a contradiction.
	It has no solution.

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:
$12c + 5\left(5 + 3c\right) = 3\left(9c - 4\right)$.

Solution:

contradiction; no solution

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:
$4\left(7d+18\right)=13\left(3d-2\right)-11d.$

Solution:

contradiction; no solution

We summarize the methods for classifying equations in the table.

Type of equation	What happens when you solve it?	Solution
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	True for any value of the variable	All real numbers
Contradiction	False for all values of the variable	No solution

Solve Equations with Fraction or Decimal Coefficients

We could use the General Strategy to solve the next example. This method would work fine, but many students do not feel very confident when they see all those fractions. So, we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

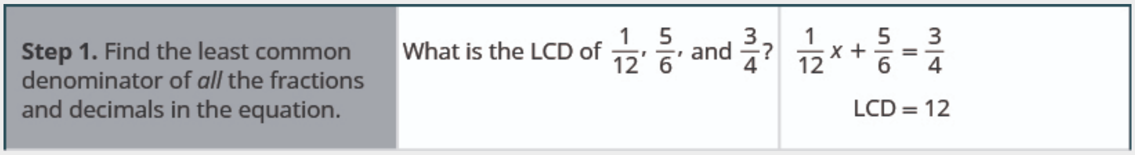
We will apply the Multiplication Property of Equality and multiply both sides of an equation by the least common denominator (LCD) of all the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but without fractions. This process is called *clearing* the equation of fractions.

To clear an equation of decimals, we think of all the decimals in their fraction form and then find the LCD of those denominators.

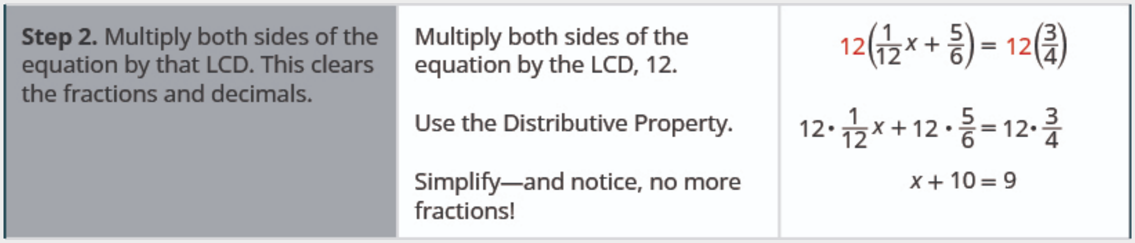
Example:
How to Solve Equations with Fraction or Decimal Coefficients
Exercise:

Problem: Solve: $\frac{1}{12}x + \frac{5}{6} = \frac{3}{4}$.

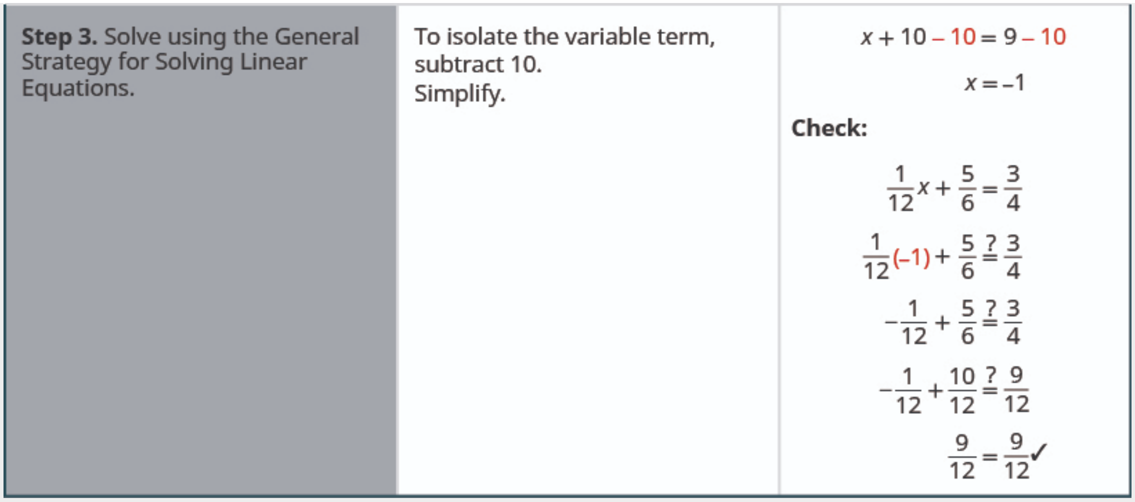
Solution:



Step 1. Find the least common denominator of *all* the fractions and decimals in the equation.	What is the LCD of $\frac{1}{12}$, $\frac{5}{6}$, and $\frac{3}{4}$?	$\frac{1}{12}x + \frac{5}{6} = \frac{3}{4}$ LCD = 12



Step 2. Multiply both sides of the equation by that LCD. This clears the fractions and decimals.	Multiply both sides of the equation by the LCD, 12.	$12\left(\frac{1}{12}x + \frac{5}{6}\right) = 12\left(\frac{3}{4}\right)$
	Use the Distributive Property.	$12 \cdot \frac{1}{12}x + 12 \cdot \frac{5}{6} = 12 \cdot \frac{3}{4}$
	Simplify—and notice, no more fractions!	$x + 10 = 9$



Step 3. Solve using the General Strategy for Solving Linear Equations.	To isolate the variable term, subtract 10. Simplify.	$x + 10 - 10 = 9 - 10$ $x = -1$ **Check:** $\frac{1}{12}x + \frac{5}{6} = \frac{3}{4}$ $\frac{1}{12}(-1) + \frac{5}{6} \overset{?}{=} \frac{3}{4}$ $-\frac{1}{12} + \frac{5}{6} \overset{?}{=} \frac{3}{4}$ $-\frac{1}{12} + \frac{10}{12} \overset{?}{=} \frac{9}{12}$ $\frac{9}{12} = \frac{9}{12} \checkmark$

Note:
Exercise:

Problem: Solve: $\frac{1}{4}x + \frac{1}{2} = \frac{5}{8}$.

Solution:

$x = \frac{1}{2}$

Note:
Exercise:

Problem: Solve: $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$.

Solution:

$x = -2$

Notice in the previous example, once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve. We then used the General Strategy for Solving Linear Equations.

Note:
Solve Equations with Fraction or Decimal Coefficients.

Find the least common denominator (LCD) of *all* the fractions and decimals (in fraction form) in the equation.
Multiply both sides of the equation by that LCD. This clears the fractions and decimals.
Solve using the General Strategy for Solving Linear Equations.

Example:
Exercise:

Problem: Solve: $5=\frac{1}{2}y+\frac{2}{3}y-\frac{3}{4}y$.

Solution:

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the LCD of all fractions in the equation.	$5=\frac{1}{2}y+\frac{2}{3}y-\frac{3}{4}y$

The LCD is 12.		
Multiply both sides of the equation by 12.		$12(5) = 12 \cdot \left(\frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y\right)$
Distribute.		$12(5) = 12 \cdot \frac{1}{2}y + 12 \cdot \frac{2}{3}y - 12 \cdot \frac{3}{4}y$
Simplify—notice, no more fractions.		$60 = 6y + 8y - 9y$
Combine like terms.		$60 = 5y$
Divide by five.		$\frac{60}{5} = \frac{5y}{5}$
Simplify.		$12 = y$
Check:	$5 = \frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y$	
Let $y = 12$.	$5 \overset{?}{=} \frac{1}{2}(12) + \frac{2}{3}(12) - \frac{3}{4}(12)$	
	$5 \overset{?}{=} 6 + 8 - 9$	
	$5 = 5 \checkmark$	

Note:
Exercise:

Problem: Solve: $7 = \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x$.

Solution:

$x = 12$

Note:
Exercise:

Problem: Solve: $-1 = \frac{1}{2}u + \frac{1}{4}u - \frac{2}{3}u$.

Solution:

$u = -12$

In the next example, we’ll distribute before we clear the fractions.

Example:
Exercise:

Problem: Solve: $\frac{1}{2}(y-5) = \frac{1}{4}(y-1)$.

Solution:

	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Distribute.	$\frac{1}{2}\cdot y-\frac{1}{2}\cdot 5=\frac{1}{4}\cdot y-\frac{1}{4}\cdot 1$
Simplify.	$\frac{1}{2}y-\frac{5}{2}=\frac{1}{4}y-\frac{1}{4}$
Multiply by the LCD, four.	$4\left(\frac{1}{2}y-\frac{5}{2}\right)=4\left(\frac{1}{4}y-\frac{1}{4}\right)$
Distribute.	$4\cdot\frac{1}{2}y-4\cdot\frac{5}{2}=4\cdot\frac{1}{4}y-4\cdot\frac{1}{4}$
Simplify.	$2y-10=y-1$
Collect the variables to the left.	$2y-y-10=y-y-1$
Simplify.	$y-10=-1$

Collect the constants to the right.	$y - 10 + 10 = -1 + 10$
Simplify.	$y = 9$

An alternate way to solve this equation is to clear the fractions without distributing first. If you multiply the factors correctly, this method will be easier.

	$\frac{1}{2}(y - 5) = \frac{1}{4}(y - 1)$
Multiply by the LCD, 4.	$4 \cdot \frac{1}{2}(y - 5) = 4 \cdot \frac{1}{4}(y - 1)$
Multiply four times the fractions.	$2(y - 5) = 1(y - 1)$
Distribute.	$2y - 10 = y - 1$
Collect the variables to the left.	$2y - y - 10 = y - y - 1$
Simplify.	$y - 10 = -1$
Collect the constants to the right.	$y - 10 + 10 = -1 + 10$

Simplify.		$y=9$
Check:	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$	
Let $y=9$.	$\frac{1}{2}(9-5)\stackrel{?}{=}\frac{1}{4}(9-1)$	
Finish the check on your own.		

Note:
Exercise:

Problem: Solve: $\frac{1}{5}(n+3)=\frac{1}{4}(n+2)$.

Solution:

$n=2$

Note:
Exercise:

Problem: Solve: $\frac{1}{2}(m-3)=\frac{1}{4}(m-7)$.

Solution:

$m=-1$

When you multiply both sides of an equation by the LCD of the fractions, make sure you multiply each term by the LCD—even if it does not contain a fraction.

Example:
Exercise:

Problem: Solve: $\frac{4q+3}{2}+6=\frac{3q+5}{4}$

Solution:

	$\frac{4q+3}{2}+6=\frac{3q+5}{4}$
Multiply both sides by the LCD, 4.	$4\left(\frac{4q+3}{2}+6\right)=4\left(\frac{3q+5}{4}\right)$
Distribute.	$4\left(\cdot\frac{4q+3}{2}\right)+4\cdot 6=4\cdot\left(\frac{3q+5}{4}\right)$
Simplify.	$2(4q+3)+24=3q+5$
	$8q+6+24=3q+5$

		$8q + 30 = 3q + 5$
Collect the variables to the left.		$8q - 3q + 30 = 3q - 3q + 5$
Simplify.		$5q + 30 = 5$
Collect the constants to the right.		$5q + 30 - 30 = 5 - 30$
Simplify.		$5q = -25$
Divide both sides by five.		$\frac{5q}{5} = \frac{-25}{5}$
Simplify.		$q = -5$
Check:	$\frac{4q+3}{2} + 6 = \frac{3q+5}{4}$	
Let $q = -5$.	$\frac{4(-5)+3}{2} + 6 \stackrel{?}{=} \frac{3(-5)+5}{4}$	
Finish the check on your own.		

Note:
Exercise:

Problem: Solve: $\frac{3r+5}{6}+1=\frac{4r+3}{3}$.

Solution:

$r=1$

Note:
Exercise:

Problem: Solve: $\frac{2s+3}{2}+1=\frac{3s+2}{4}$.

Solution:

$s=-8$

Some equations have decimals in them. This kind of equation may occur when we solve problems dealing with money or percentages. But decimals can also be expressed as fractions. For example, $0.7=\frac{7}{10}$ and $0.29=\frac{29}{100}$. So, with an equation with decimals, we can use the same method we used to clear fractions—multiply both sides of the equation by the least common denominator.

The next example uses an equation that is typical of the ones we will see in the money applications in a later section. Notice that we will clear all decimals by multiplying by the LCD of their fraction form.

Example:

Exercise:

Problem: Solve: $0.25x + 0.05(x + 3) = 2.85$.

Solution:

Look at the decimals and think of the equivalent fractions:
Equation:

$$0.25 = \frac{25}{100}, \qquad 0.05 = \frac{5}{100}, \qquad 2.85 = 2\frac{85}{100}.$$

Notice, the LCD is 100. By multiplying by the LCD we will clear the decimals from the equation.

	$0.25x + 0.05(x + 3) = 2.85$
Distribute first.	$0.25x + 0.05x + 0.15 = 2.85$
Combine like terms.	$0.30x + 0.15 = 2.85$
To clear decimals, multiply by 100.	$100(0.30x + 0.15) = 100(2.85)$
Distribute.	$30x + 15 = 285$

Subtract 15 from both sides.	$30x + 15 - 15 = 285 - 15$
Simplify.	$30x = 270$
Divide by 30.	$\frac{30x}{30} = \frac{270}{30}$
Simplify.	$x = 9$
Check it yourself by substituting $x = 9$ into the original equation.	

Note:
Exercise:

Problem: Solve: $0.25n + 0.05(n + 5) = 2.95$.

Solution:

$n = 9$

Note:
Exercise:

Problem: Solve: $0.10d + 0.05(d - 5) = 2.15$.

Solution:

$d = 16$

Key Concepts

- **How to determine whether a number is a solution to an equation**

  Substitute the number in for the variable in the equation.
  Simplify the expressions on both sides of the equation.
  Determine whether the resulting equation is true. If it is true, the number is a solution. If it is not true, the number is not a solution.

- **How to Solve Linear Equations Using a General Strategy**

  Simplify each side of the equation as much as possible. Use the Distributive Property to remove any parentheses. Combine like terms.
  Collect all the variable terms on one side of the equation. Use the Addition or Subtraction Property of Equality.
  Collect all the constant terms on the other side of the equation. Use the Addition or Subtraction Property of Equality.
  Make the coefficient of the variable term equal to 1. Use the Multiplication or Division Property of Equality. State the solution to the equation.
  Check the solution. Substitute the solution into the original equation to make sure the result is a true statement.

- **How to Solve Equations with Fraction or Decimal Coefficients**

  Find the least common denominator (LCD) of *all* the fractions and decimals (in fraction form) in the equation.
  Multiply both sides of the equation by that LCD. This clears the fractions and decimals.
  Solve using the General Strategy for Solving Linear Equations.

Practice Makes Perfect

Solve Equations Using the General Strategy

In the following exercises, determine whether the given values are solutions to the equation.

Exercise:

Problem: $6y + 10 = 12y$

ⓐ $y = \frac{5}{3}$
ⓑ $y = -\frac{1}{2}$

Solution:

ⓐ yes ⓑ no

Exercise:

Problem: $4x + 9 = 8x$

ⓐ $x = -\frac{7}{8}$
ⓑ $x = \frac{9}{4}$

Exercise:

Problem: $8u - 1 = 6u$

ⓐ $u = -\frac{1}{2}$
ⓑ $u = \frac{1}{2}$

Solution:

ⓐ no ⓑ yes

Exercise:

Problem: $9v - 2 = 3v$

ⓐ $v = -\frac{1}{3}$
ⓑ $v = \frac{1}{3}$

In the following exercises, solve each linear equation.

Exercise:

Problem: $15\left(y - 9\right) = -60$

Solution:

$y = 5$

Exercise:

Problem: $-16\left(3n + 4\right) = 32$

Exercise:

Problem: $-\left(w - 12\right) = 30$

Solution:

$w = -18$

Exercise:

Problem: $-\left(t - 19\right) = 28$

Exercise:

Problem: $51 + 5(4 - q) = 56$

Solution:

$q = 3$

Exercise:

Problem: $-6 + 6(5 - k) = 15$

Exercise:

Problem: $3(10 - 2x) + 54 = 0$

Solution:

$x = 14$

Exercise:

Problem: $-2(11 - 7x) + 54 = 4$

Exercise:

Problem: $\frac{2}{3}(9c - 3) = 22$

Solution:

$c = 4$

Exercise:

Problem: $\frac{3}{5}(10x - 5) = 27$

Exercise:

Problem: $\frac{1}{5}(15c + 10) = c + 7$

Solution:

$c = \frac{5}{2}$

Exercise:

Problem: $\frac{1}{4}(20d+12)=d+7$

Exercise:

Problem: $3(4n-1)-2=8n+3$

Solution:

$n=2$

Exercise:

Problem: $9(2m-3)-8=4m+7$

Exercise:

Problem: $12+2(5-3y)=-9\left(y-1\right)-2$

Solution:

$y=-5$

Exercise:

Problem: $-15+4(2-5y)=-7\left(y-4\right)+4$

Exercise:

Problem: $5+6(3s-5)=-3+2\left(8s-1\right)$

Solution:

$s=10$

Exercise:

Problem: $-12+8(x-5)=-4+3\left(5x-2\right)$

Exercise:

Problem: $4\left(p-4\right)-\left(p+7\right)=5\left(p-3\right)$

Solution:

$p=-4$

Exercise:

Problem: $3\left(a-2\right)-\left(a+6\right)=4\left(a-1\right)$

Exercise:

Problem: $4\left[5-8\left(4c-3\right)\right]=12\left(1-13c\right)-8$

Solution:

$c=-4$

Exercise:

Problem: $5\left[9-2\left(6d-1\right)\right]=11\left(4-10d\right)-139$

Exercise:

Problem: $3\left[-9+8\left(4h-3\right)\right]=2\left(5-12h\right)-19$

Solution:

$h=\frac{3}{4}$

Exercise:

Problem: $3\left[-14+2\left(15k-6\right)\right]=8\left(3-5k\right)-24$

Exercise:

Problem: $5\left[2\left(m+4\right)+8\left(m-7\right)\right]=2\left[3\left(5+m\right)-\left(21-3m\right)\right]$

Solution:

$m=6$

Exercise:

Problem: $10\left[5\left(n+1\right)+4\left(n-1\right)\right]=11\left[7\left(5+n\right)-\left(25-3n\right)\right]$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

Exercise:

Problem: $23z+19=3\left(5z-9\right)+8z+46$

Solution:

identity; all real numbers

Exercise:

Problem: $15y+32=2\left(10y-7\right)-5y+46$

Exercise:

Problem: $18(5j-1)+29=47$

Solution:

conditional equation;$j=\frac{2}{5}$

Exercise:

Problem: $24(3d-4)+100=52$

Exercise:

Problem: $22(3m-4)=8\left(2m+9\right)$

Solution:

conditional equation; $m=\frac{16}{5}$

Exercise:

Problem: $30(2n-1)=5\left(10n+8\right)$

Exercise:

Problem: $7v+42=11\left(3v+8\right)-2\left(13v-1\right)$

Solution:

contradiction; no solution

Exercise:

Problem: $18u-51=9\left(4u+5\right)-6\left(3u-10\right)$

Exercise:

Problem: $45(3y-2)=9\left(15y-6\right)$

Solution:

contradiction; no solution

Exercise:

Problem: $60(2x-1)=15\left(8x+5\right)$

Exercise:

Problem: $9(14d+9)+4d=13\left(10d+6\right)+3$

Solution:

identity; all real numbers

Exercise:

Problem: $11(8c+5)-8c=2\left(40c+25\right)+5$

Solve Equations with Fraction or Decimal Coefficients

In the following exercises, solve each equation with fraction coefficients.

Exercise:

Problem: $\frac{1}{4}x-\frac{1}{2}=-\frac{3}{4}$

Solution:

$x=-1$

Exercise:

Problem: $\frac{3}{4}x-\frac{1}{2}=\frac{1}{4}$

Exercise:

Problem: $\frac{5}{6}y-\frac{2}{3}=-\frac{3}{2}$

Solution:

$y=-1$

Exercise:

Problem: $\frac{5}{6}y - \frac{1}{3} = -\frac{7}{6}$

Exercise:

Problem: $\frac{1}{2}a + \frac{3}{8} = \frac{3}{4}$

Solution:

$a = \frac{3}{4}$

Exercise:

Problem: $\frac{5}{8}b + \frac{1}{2} = -\frac{3}{4}$

Exercise:

Problem: $2 = \frac{1}{3}x - \frac{1}{2}x + \frac{2}{3}x$

Solution:

$x = 4$

Exercise:

Problem: $2 = \frac{3}{5}x - \frac{1}{3}x + \frac{2}{5}x$

Exercise:

Problem: $\frac{1}{3}w + \frac{5}{4} = w - \frac{1}{4}$

Solution:

$w = \frac{9}{4}$

Exercise:

Problem: $\frac{1}{2}a - \frac{1}{4} = \frac{1}{6}a + \frac{1}{12}$

Exercise:

Problem: $\frac{1}{3}b + \frac{1}{5} = \frac{2}{5}b - \frac{3}{5}$

Solution:

$b = 12$

Exercise:

Problem: $\frac{1}{3}x + \frac{2}{5} = \frac{1}{5}x - \frac{2}{5}$

Exercise:

Problem: $\frac{1}{4}(p - 7) = \frac{1}{3}(p + 5)$

Solution:

$p = -41$

Exercise:

Problem: $\frac{1}{5}(q + 3) = \frac{1}{2}(q - 3)$

Exercise:

Problem: $\frac{1}{2}(x + 4) = \frac{3}{4}$

Solution:

$x = -\frac{5}{2}$

Exercise:

Problem: $\frac{1}{3}(x+5)=\frac{5}{6}$

Exercise:

Problem: $\frac{4n+8}{4}=\frac{n}{3}$

Solution:

$n=-3$

Exercise:

Problem: $\frac{3p+6}{3}=\frac{p}{2}$

Exercise:

Problem: $\frac{3x+4}{2}+1=\frac{5x+10}{8}$

Solution:

$x=-2$

Exercise:

Problem: $\frac{10y-2}{3}+3=\frac{10y+1}{9}$

Exercise:

Problem: $\frac{7u-1}{4}-1=\frac{4u+8}{5}$

Solution:

$u=3$

Exercise:

Problem: $\frac{3v-6}{2}+5=\frac{11v-4}{5}$

In the following exercises, solve each equation with decimal coefficients.

Exercise:

Problem: $0.4x+0.6=0.5x-1.2$

Solution:

$x=18$

Exercise:

Problem: $0.7x+0.4=0.6x+2.4$

Exercise:

Problem: $0.9x-1.25=0.75x+1.75$

Solution:

$x=20$

Exercise:

Problem: $1.2x-0.91=0.8x+2.29$

Exercise:

Problem: $0.05n+0.10\left(n+8\right)=2.15$

Solution:

$n=9$

Exercise:

Problem: $0.05n + 0.10\left(n + 7\right) = 3.55$

Exercise:

Problem: $0.10d + 0.25\left(d + 5\right) = 4.05$

Solution:

$d = 8$

Exercise:

Problem: $0.10d + 0.25\left(d + 7\right) = 5.25$

Everyday Math

Exercise:

Problem:

Fencing Micah has 74 feet of fencing to make a dog run in his yard. He wants the length to be 2.5 feet more than the width. Find the length, L, by solving the equation $2L + 2(L - 2.5) = 74$.

Solution:

$L = 19.75$ feet

Exercise:

Problem:

Stamps Paula bought \$22.82 worth of 49-cent stamps and 21-cent stamps. The number of 21-cent stamps was eight less than the number of
49-cent stamps. Solve the equation $0.49s+0.21(s-8)=22.82$ for s, to find the number of 49-cent stamps Paula bought.

Writing Exercises

Exercise:

Problem:

Using your own words, list the steps in the general strategy for solving linear equations.

Solution:

Answers will vary.

Exercise:

Problem:

Explain why you should simplify both sides of an equation as much as possible before collecting the variable terms to one side and the constant terms to the other side.

Exercise:

Problem:

What is the first step you take when solving the equation $3-7(y-4)=38$? Why is this your first step?

Solution:

Answers will vary.

Exercise:

Problem:

If an equation has several fractions, how does multiplying both sides by the LCD make it easier to solve?

Exercise:

Problem:

If an equation has fractions only on one side, why do you have to multiply both sides of the equation by the LCD?

Solution:

Answers will vary.

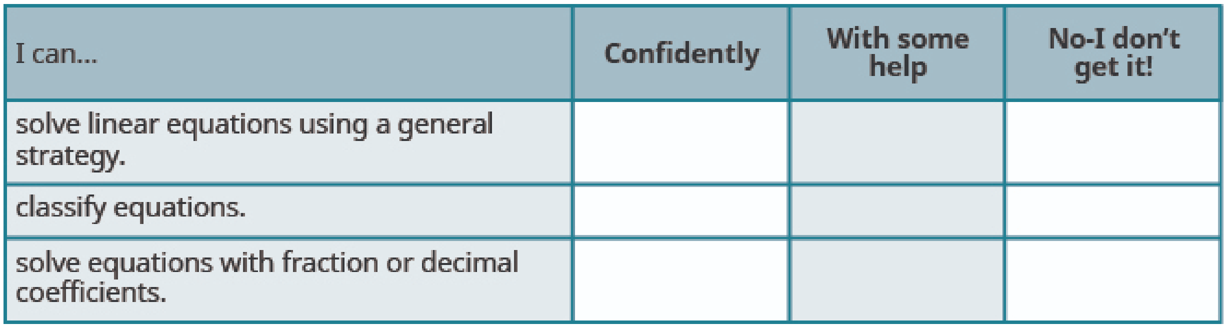
Exercise:

Problem:

For the equation $0.35x + 2.1 = 3.85$, how do you clear the decimal?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
solve linear equations using a general strategy.			
classify equations.			
solve equations with fraction or decimal coefficients.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

conditional equation

An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

contradiction

An equation that is false for all values of the variable is called a contradiction. A contradiction has no solution.

identity

An equation that is true for any value of the variable is called an Identity. The solution of an identity is all real numbers.

linear equation

A linear equation is an equation in one variable that can be written, where a and b are real numbers and $a \neq 0$, as $ax + b = 0$.

solution of an equation

A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

Solve a Formula for a Specific Variable

By the end of this section, you will be able to:

- Use the distance, rate, and time formula
- Solve a formula for a specific variable

Note:
Before you get started, take this readiness quiz.

1. Write 35 miles per gallon as a unit rate.
   If you missed this problem, review [link].
2. Solve $6x+24=96$.
   If you missed this problem, review [link].
3. Find the simple interest earned after 5 years on $\$1,000$ at an interest rate of 4%.
   If you missed this problem, review [link].

Use the Distance, Rate, and Time Formula

One formula you'll use often in algebra and in everyday life is the formula for distance traveled by an object moving at a constant speed. The basic idea is probably already familiar to you. Do you know what distance you travel if you drove at a steady rate of 60 miles per hour for 2 hours? (This might happen if you use your car's cruise control while driving on the Interstate.) If you said 120 miles, you already know how to use this formula!

The math to calculate the distance might look like this:
Equation:

$$\begin{aligned}\text{distance} &= \left(\frac{60\text{ miles}}{1\text{ hour}}\right)(2\text{ hours})\\ \text{distance} &= 120\text{ miles}\end{aligned}$$

In general, the formula relating distance, rate, and time is

Equation:

$$\text{distance} = \text{rate} \cdot \text{time}$$

Note:

Distance, Rate and Time

For an object moving in at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula

Equation:

$$d = rt$$

where $d = \text{distance}$, $r = \text{rate}$, and $t = \text{time}$.

Notice that the units we used above for the rate were miles per hour, which we can write as a ratio $\frac{miles}{hour}$. Then when we multiplied by the time, in hours, the common units ‘hour’ divided out. The answer was in miles.

Example:

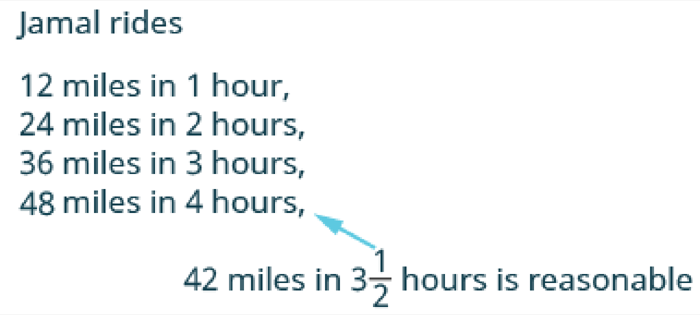
Exercise:

Problem:

Jamal rides his bike at a uniform rate of 12 miles per hour for $3\frac{1}{2}$ hours. How much distance has he traveled?

Solution:

Solution

Step 1. **Read** the problem. You may want to create a mini-chart to summarize the information in the problem.	$d = ?$ $r = 12\text{ mph}$ $t = 3\frac{1}{2}\text{ hours}$
Step 2. **Identify** what you are looking for.	distance traveled
Step 3. **Name.** Choose a variable to represent it.	let *d* = distance
Step 4. **Translate.** Write the appropriate formula for the situation. Substitute in the given information.	$d = rt$ $d = 12 \cdot 3\frac{1}{2}$
Step 5. **Solve** the equation.	$d = 42\text{ miles}$
Step 6. **Check:** Does 42 miles make sense?  Jamal rides 12 miles in 1 hour, 24 miles in 2 hours, 36 miles in 3 hours, 48 miles in 4 hours, 42 miles in $3\frac{1}{2}$ hours is reasonable	
Step 7. **Answer** the question with a complete sentence.	Jamal rode 42 miles.

Note:

Exercise:

Problem:

Lindsay drove for $5\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?

Solution:

330 mi

Note:
Exercise:

Problem:

Trinh walked for $2\frac{1}{3}$ hours at 3 miles per hour. How far did she walk?

Solution:

7 mi

Example:
Exercise:

Problem:

Rey is planning to drive from his house in San Diego to visit his grandmother in Sacramento, a distance of 520 miles. If he can drive at a steady rate of 65 miles per hour, how many hours will the trip take?

Solution:
Solution

Step 1. **Read** the problem. Summarize the information in the problem.	$d = 520\ \text{miles}$ $r = 65\ \text{mph}$ $t = ?$
Step 2. **Identify** what you are looking for.	how many hours (time)
Step 3. **Name:** Choose a variable to represent it.	let t = time
Step 4. **Translate.** Write the appropriate formula. Substitute in the given information.	$d = rt$ $520 = 65t$
Step 5. **Solve** the equation.	$t = 8$
Step 6. **Check:** Substitute the numbers into the formula and make sure the result is a true statement. $d = rt$ $520 \overset{?}{=} 65 \cdot 8$ $520 = 520 > \checkmark$	
Step 7. **Answer** the question with a complete sentence. We know the units of time will be hours because we divided miles by miles per hour.	Rey's trip will take 8 hours.

Note:
Exercise:

Problem:

Lee wants to drive from Phoenix to his brother's apartment in San Francisco, a distance of 770 miles. If he drives at a steady rate of 70 miles per hour, how many hours will the trip take?

Solution:

11 hours

Note:
Exercise:

Problem:

Yesenia is 168 miles from Chicago. If she needs to be in Chicago in 3 hours, at what rate does she need to drive?

Solution:

56 mph

Solve a Formula for a Specific Variable

In this chapter, you became familiar with some formulas used in geometry. Formulas are also very useful in the sciences and social sciences—fields such as chemistry, physics, biology, psychology, sociology, and criminal justice. Healthcare workers use formulas, too, even for something as routine as dispensing medicine. The widely used spreadsheet program Microsoft Excel^{TM} relies on formulas to do its calculations. Many teachers use spreadsheets to apply formulas to compute student grades. It is important to be familiar with formulas and be able to manipulate them easily.

In [link] and [link], we used the formula $d = rt$. This formula gives the value of d when you substitute in the values of r and t. But in [link], we had to find the value of t. We substituted in values of d and r and then used algebra to solve to t. If you had to do this often, you might wonder why there isn't a formula that gives the value of t when you substitute in the values of d and r. We can get a formula like this by solving the formula $d = rt$ for t.

To solve a formula for a specific variable means to get that variable by itself with a coefficient of 1 on one side of the equation and all the other variables and constants on the other side. We will call this solving an equation for a specific variable *in general*. This process is also called *solving a literal equation*. The result is another formula, made up only of variables. The formula contains letters, or *literals*.

Let's try a few examples, starting with the distance, rate, and time formula we used above.

Example:
Exercise:

Problem: Solve the formula $d = rt$ for t:

ⓐ when $d = 520$ and $r = 65$
ⓑ in general.

Solution:
Solution

We'll write the solutions side-by-side so you can see that solving a formula in general uses the same steps as when we have numbers to substitute.

	ⓐ when $d = 520$ and $r = 65$	ⓑ in general
Write the forumla.	$d = rt$	$d = rt$
Substitute any given values.	$520 = 65t$	
Divide to isolate *t*.	$\frac{520}{65} = \frac{65t}{65}$	$\frac{d}{r} = \frac{rt}{r}$
Simplify.	$8 = t$ $t = 8$	$\frac{d}{r} = t$ $t = \frac{d}{r}$

Notice that the solution for ⓐ is the same as that in [link]. We say the formula $t = \frac{d}{r}$ is solved for t. We can use this version of the formula anytime we are given the distance and rate and need to find the time.

Note:
Exercise:

Problem: Solve the formula $d = rt$ for r:

ⓐ when $d = 180$ and $t = 4$
ⓐ in general

Solution:

ⓐ $r = 45$
ⓑ $r = \frac{d}{t}$

Note:
Exercise:

Problem: Solve the formula $d = rt$ for r:

ⓐ when $d = 780$ and $t = 12$
ⓑ in general

Solution:

ⓐ $r = 65$
ⓑ $r = \frac{d}{t}$

We used the formula $A = \frac{1}{2}bh$ in Use Properties of Rectangles, Triangles, and Trapezoids to find the area of a triangle when we were given the base and height. In the next example, we will solve this formula for the height.

Example:
Exercise:

Problem:

The formula for area of a triangle is $A = \frac{1}{2}bh$. Solve this formula for h:

ⓐ when $A = 90$ and $b = 15$
ⓑ in general

Solution:
Solution

	ⓐ when $A = 90$ and $b = 15$	ⓑ in general
Write the forumla.	$A = \frac{1}{2}bh$	$A = \frac{1}{2}bh$
Substitute any given values.	$90 = \frac{1}{2} \cdot 15 \cdot h$	
Clear the fractions.	$2 \cdot 90 = 2 \cdot \frac{1}{2} \cdot 15 \cdot h$	$2 \cdot A = 2 \cdot \frac{1}{2} \cdot b \cdot h$
Simplify.	$180 = 15h$	$2A = bh$
Solve for h.	$12 = h$	$\frac{2A}{b} = h$

We can now find the height of a triangle, if we know the area and the base, by using the formula

Equation:

$$h = \frac{2A}{b}$$

Note:
Exercise:

Problem: Use the formula $A = \frac{1}{2}bh$ to solve for h:

ⓐ when $A = 170$ and $b = 17$
ⓑ in general

Solution:

ⓐ $h = 20$ ⓑ $h = \frac{2A}{b}$

Note:
Exercise:

Problem: Use the formula $A = \frac{1}{2}bh$ to solve for b:

ⓐ when $A = 62$ and $h = 31$
ⓑ in general

Solution:

ⓐ $b = 4$
ⓑ $b = \frac{2A}{h}$

In Solve Simple Interest Applications, we used the formula $I = Prt$ to calculate simple interest, where I is interest, P is principal, r is rate as a decimal, and t is time in years.

Example:
Exercise:

Problem: Solve the formula $I = Prt$ to find the principal, P:

ⓐ when $I = \$5{,}600,\ r = 4\%,\ t = 7$ years
ⓑ in general

Solution:
Solution

	I = \$5600, r = 4%, t = 7 years	in general
Write the forumla.	$I = Prt$	$I = Prt$
Substitute any given values.	$5600 = P(0.04)(7)$	$I = Prt$

Multiply $r \cdot t$.	$5600 = P(0.28)$	$I = P(rt)$
Divide to isolate P.	$\frac{5600}{0.28} = \frac{P(0.28)}{0.28}$	$\frac{I}{rt} = \frac{P(rt)}{rt}$
Simplify.	$20{,}000 = P$	$\frac{I}{rt} = P$
State the answer.	The principal is $20,000.	$P = \frac{I}{rt}$

Note:
Exercise:

Problem: Use the formula $I = Prt$.

Find t: ⓐ when $I = \$2{,}160,\ r = 6\%,\ P = \$12{,}000$; ⓑ in general

Solution:

ⓐ $t = 3$ years
ⓑ $t = \frac{I}{Pr}$

Note:
Exercise:

Problem: Use the formula $I = Prt$.

Find r: ⓐ when $I = \$5{,}400,\ P = \$9{,}000,\ t = 5$ years ⓑ in general

Solution:

ⓐ $r = 0.12 = \ 12\%$
ⓑ $r = \frac{I}{Pt}$

Later in this class, and in future algebra classes, you'll encounter equations that relate two variables, usually x and y. You might be given an equation that is solved for y and need to solve it for x, or vice versa. In the following example, we're given an equation with both x and y on the same side and we'll solve it for y. To do this, we will follow the same steps that we used to solve a formula for a specific variable.

Example:
Exercise:

Problem: Solve the formula $3x + 2y = 18$ for y:

ⓐ when $x = 4$
ⓑ in general

Solution:
Solution

	when $x = 4$	in general
Write the equation.	$3x + 2y = 18$	$3x + 2y = 18$
Substitute any given values.	$3(4) + 2y = 18$	$3x + 2y = 18$
Simplify if possible.	$12 + 2y = 18$	$3x + 2y = 18$
Subtract to isolate the *y*-term.	$12 - 12 + 2y = 18 - 12$	$3x - 3x + 2y = 18 - 3x$
Simplify.	$2y = 6$	$2y = 18 - 3x$
Divide.	$\frac{2y}{2} = \frac{6}{2}$	$\frac{2y}{2} = \frac{18 - 3x}{2}$
Simplify.	$y = 3$	$y = \frac{18 - 3x}{2}$

Note:
Exercise:

Problem: Solve the formula $3x + 4y = 10$ for y:

ⓐ when $x=2$
ⓑ in general

Solution:

ⓐ $y=1$
ⓑ $y=\frac{10-3x}{4}$

Note:
Exercise:

Problem: Solve the formula $5x+2y=18$ for y:

ⓐ when $x=4$
ⓑ in general

Solution:

ⓐ $y=-1$
ⓑ $y=\frac{18-5x}{2}$

In the previous examples, we used the numbers in part (a) as a guide to solving in general in part (b). Do you think you're ready to solve a formula in general without using numbers as a guide?

Example:
Exercise:

Problem: Solve the formula $P = a + b + c$ for a.

Solution:
Solution

We will isolate a on one side of the equation.

We will isolate *a* on one side of the equation.		
Write the equation.		$P = a + b + c$
Subtract *b* and *c* from both sides to isolate *a*.		$P - b - c = a + b + c - b - c$
Simplify.		$P - b - c = a$

So, $a = P - b - c$

Note:
Exercise:

Problem: Solve the formula $P = a + b + c$ for b.

Solution:

$b = P - a - c$

Note:
Exercise:

Problem: Solve the formula $P = a + b + c$ for c.

Solution:

$c = P - a - b$

Example:
Exercise:

Problem: Solve the equation $3x + y = 10$ for y.

Solution:
Solution

We will isolate y on one side of the equation.

We will isolate y on one side of the equation.		
Write the equation.		$3x + y = 10$
Subtract $3x$ from both sides to isolate y.		$3x - 3x + y = 10 - 3x$
Simplify.		$y = 10 - 3x$

Note:
Exercise:

Problem: Solve the formula $7x + y = 11$ for y.

Solution:

$y = 11 - 7x$

Note:
Exercise:

Problem: Solve the formula $11x + y = 8$ for y.

Solution:

$y = 8 - 11x$

Example:
Exercise:

Problem: Solve the equation $6x + 5y = 13$ for y.

Solution:
Solution

We will isolate y on one side of the equation.

We will isolate *y* on one side of the equation.	
Write the equation.	$6x + 5y = 13$
Subtract to isolate the term with *y*.	$6x + 5y - 6x = 13 - 6x$
Simplify.	$5y = 13 - 6x$
Divide 5 to make the coefficient 1.	$\frac{5y}{5} = \frac{13 - 6x}{5}$
Simplify.	$y = \frac{13 - 6x}{5}$

Note:
Exercise:

Problem: Solve the formula $4x + 7y = 9$ for y.

Solution:

$y = \frac{9-4x}{7}$

Note:
Exercise:

Problem: Solve the formula $5x + 8y = 1$ for y.

Solution:

$y = \frac{1-5x}{8}$

Note: The Links to Literacy activity *What's Faster than a Speeding Cheetah?* will provide you with another view of the topics covered in this section.

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Distance $=$ Rate x Time
- Distance, Rate, Time
- Simple Interest
- Solving a Formula for a Specific Variable
- Solving a Formula for a Specific Variable

Key Concepts

- **Distance, Rate, and Time**
  - $d = rt$

Section Exercises

Practice Makes Perfect

Use the Distance, Rate, and Time Formula

In the following exercises, solve.

Exercise:

Problem:

Steve drove for $8\frac{1}{2}$ hours at 72 miles per hour. How much distance did he travel?

Solution:

612 mi

Exercise:

Problem:

Socorro drove for $4\frac{5}{6}$ hours at 60 miles per hour. How much distance did she travel?

Exercise:

Problem:

Yuki walked for $1\frac{3}{4}$ hours at 4 miles per hour. How far did she walk?

Solution:

7 mi

Exercise:

Problem:

Francie rode her bike for $2\frac{1}{2}$ hours at 12 miles per hour. How far did she ride?

Exercise:

Problem:

Connor wants to drive from Tucson to the Grand Canyon, a distance of 338 miles. If he drives at a steady rate of 52 miles per hour, how many hours will the trip take?

Solution:

6.5 hours

Exercise:

Problem:

Megan is taking the bus from New York City to Montreal. The distance is 384 miles and the bus travels at a steady rate of 64 miles per hour. How long will the bus ride be?

Exercise:

Problem:

Aurelia is driving from Miami to Orlando at a rate of 65 miles per hour. The distance is 235 miles. To the nearest tenth of an hour, how long will the trip take?

Solution:

3.6 hours

Exercise:

Problem:

Kareem wants to ride his bike from St. Louis, Missouri to Champaign, Illinois. The distance is 180 miles. If he rides at a steady rate of 16 miles per hour, how many hours will the trip take?

Exercise:

Problem:

Javier is driving to Bangor, Maine, which is 240 miles away from his current location. If he needs to be in Bangor in 4 hours, at what rate does he need to drive?

Solution:

60 mph

Exercise:

Problem:

Alejandra is driving to Cincinnati, Ohio, 450 miles away. If she wants to be there in 6 hours, at what rate does she need to drive?

Exercise:

Problem:

Aisha took the train from Spokane to Seattle. The distance is 280 miles, and the trip took 3.5 hours. What was the speed of the train?

Solution:

80 mph

Exercise:

Problem:

Philip got a ride with a friend from Denver to Las Vegas, a distance of 750 miles. If the trip took 10 hours, how fast was the friend driving?

Solve a Formula for a Specific Variable

In the following exercises, use the formula. $d = rt$.

Exercise:

Problem: Solve for t:

ⓐ when $d = 350$ and $r = 70$
ⓑ in general

Solution:

ⓐ $t = 5$
ⓑ $t = \frac{d}{r}$

Exercise:

Problem: Solve for t:

ⓐ when $d = 240$ and $r = 60$
ⓑ in general

Exercise:

Problem: Solve for t:

ⓐ when $d = 510$ and $r = 60$
ⓑ in general

Solution:

ⓐ $t = 8.5$
ⓑ $t = \frac{d}{r}$

Exercise:

Problem: Solve for t:

ⓐ when $d = 175$ and $r = 50$
ⓑ in general

Exercise:

Problem: Solve for r:

ⓐ when $d = 204$ and $t = 3$
ⓑ in general

Solution:

ⓐ $r = 68$
ⓑ $r = \frac{d}{t}$

Exercise:

Problem: Solve for r:

ⓐ when $d = 420$ and $t = 6$
ⓑ in general

Exercise:

Problem: Solve for r:

ⓐ when $d = 160$ and $t = 2.5$
ⓑ in general

Solution:

ⓐ $r = 64$
ⓑ $r = \frac{d}{t}$

Exercise:

Problem: Solve for r:

ⓐ when $d = 180$ and $t = 4.5$
ⓑ in general.

In the following exercises, use the formula $A = \frac{1}{2}bh$.

Exercise:

Problem: Solve for b:

ⓐ when $A = 126$ and $h = 18$
ⓑ in general

Solution:

ⓐ $b = 14$
ⓑ $b = \frac{2A}{h}$

Exercise:

Problem: Solve for h:

ⓐ when $A = 176$ and $b = 22$
ⓑ in general

Exercise:

Problem: Solve for h:

ⓐ when $A = 375$ and $b = 25$
ⓑ in general

Solution:

ⓐ $h = 30$
ⓑ $h = \frac{2A}{b}$

Exercise:

Problem: Solve for b:

ⓐ when $A = 65$ and $h = 13$
ⓑ in general

In the following exercises, use the formula $I = Prt$.
Exercise:

Problem: Solve for the principal, P for:

ⓐ $I = \$5{,}480$, $r = 4\%$, $t = 7$ years
ⓑ in general

Solution:

ⓐ $P = \$19{,}571.43$
ⓑ $P = \frac{I}{rt}$

Exercise:

Problem: Solve for the principal, P for:

ⓐ $I = \$3{,}950$, $r = 6\%$, $t = 5$ years
ⓑ in general

Exercise:

Problem: Solve for the time, t for:

ⓐ $I = \$2{,}376,\ P = \$9{,}000,\ r = 4.4\%$
ⓑ in general

Solution:

ⓐ $t = 6$ years
ⓑ $t = \frac{I}{Pr}$

Exercise:

Problem: Solve for the time, t for:

ⓐ $I = \$624,\ P = \$6{,}000,\ r = 5.2\%$
ⓑ in general

In the following exercises, solve.

Exercise:

Problem: Solve the formula $2x + 3y = 12$ for y:

ⓐ when $x = 3$
ⓑ in general

Solution:

ⓐ $y = 2$
ⓑ $y = \frac{12-2x}{3}$

Exercise:

Problem: Solve the formula $5x + 2y = 10$ for y:

ⓐ when $x = 4$
ⓑ in general

Exercise:

Problem: Solve the formula $3x + y = 7$ for y:

ⓐ when $x = -2$
ⓑ in general

Solution:

ⓐ $y = 13$
ⓑ $y = 7 - 3x$

Exercise:

Problem: Solve the formula $4x + y = 5$ for y:

ⓐ when $x = -3$
ⓑ in general

Exercise:

Problem: Solve $a + b = 90$ for b.

Solution:

ⓐ $b = 90 - a$
ⓑ $a = 90 - b$

Exercise:

Problem: Solve $a + b = 90$ for a.

Exercise:

Problem: Solve $180 = a + b + c$ for a.

Solution:

$a = 180 - b - c$

Exercise:

Problem: Solve $180 = a + b + c$ for c.

Exercise:

Problem: Solve the formula $8x + y = 15$ for y.

Solution:

$y = 15 - 8x$

Exercise:

Problem: Solve the formula $9x + y = 13$ for y.

Exercise:

Problem: Solve the formula $-4x + y = -6$ for y.

Solution:

$y = -6 + 4x$

Exercise:

Problem: Solve the formula $-5x + y = -1$ for y.

Exercise:

Problem: Solve the formula $4x + 3y = 7$ for y.

Solution:

$y = \frac{7-4x}{3}$

Exercise:

Problem: Solve the formula $3x + 2y = 11$ for y.

Exercise:

Problem: Solve the formula $x - y = -4$ for y.

Solution:

$y = 4 + x$

Exercise:

Problem: Solve the formula $x - y = -3$ for y.

Exercise:

Problem: Solve the formula $P = 2L + 2W$ for L.

Solution:

$L = \frac{P-2W}{2}$

Exercise:

Problem: Solve the formula $P = 2L + 2W$ for W.

Exercise:

Problem: Solve the formula $C = \pi d$ for d.

Solution:

$d = \frac{C}{\pi}$

Exercise:

Problem: Solve the formula $C = \pi d$ for π.

Exercise:

Problem: Solve the formula $V = LWH$ for L.

Solution:

$L = \frac{V}{WH}$

Exercise:

Problem: Solve the formula $V = LWH$ for H.

Everyday Math

Exercise:

Problem:

Converting temperature While on a tour in Greece, Tatyana saw that the temperature was 40° Celsius. Solve for F in the formula $C = \frac{5}{9}(F - 32)$ to find the temperature in Fahrenheit.

Solution:

104° F

Exercise:

Problem:

Converting temperature Yon was visiting the United States and he saw that the temperature in Seattle was 50° Fahrenheit. Solve for C in the formula $F = \frac{9}{5}C + 32$ to find the temperature in Celsius.

Writing Exercises

Exercise:

Problem: Solve the equation $2x + 3y = 6$ for y:

ⓐ when $x = -3$
ⓑ in general
ⓒ Which solution is easier for you? Explain why.

Solution:

Answers will vary

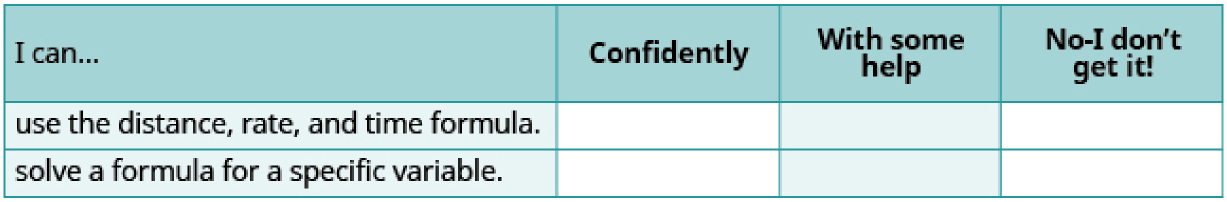
Exercise:

Problem: Solve the equation $5x - 2y = 10$ for x:

ⓐ when $y = 10$
ⓑ in general
ⓒ Which solution is easier for you? Explain why.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
use the distance, rate, and time formula.			
solve a formula for a specific variable.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next Chapter? Why or why not?

Chapter Review Exercises

Use a Problem Solving Strategy

Approach Word Problems with a Positive Attitude

In the following exercises, solve.

Exercise:

Problem:

How has your attitude towards solving word problems changed as a result of working through this chapter? Explain.

Solution:

Answers will vary.

Exercise:

Problem:

Did the Problem Solving Strategy help you solve word problems in this chapter? Explain.

Use a Problem Solving Strategy for Word Problems

In the following exercises, solve using the problem-solving strategy for word problems. Remember to write a complete sentence to answer each question.

Exercise:

Problem:

Three-fourths of the people at a concert are children. If there are 87 children, what is the total number of people at the concert?

Solution:

There are 116 people at the concert.

Exercise:

Problem:

There are 9 saxophone players in the band. The number of saxophone players is one less than twice the number of tuba players. Find the number of tuba players.

Exercise:

Problem:

Reza was very sick and lost 15% of his original weight. He lost 27 pounds. What was his original weight?

Solution:

His original weight was 180 pounds.

Exercise:

Problem:

Dolores bought a crib on sale for $\$350$. The sale price was 40% of the original price. What was the original price of the crib?

Solve Number Problems

In the following exercises, solve each number word problem.

Exercise:

Problem:

The sum of a number and three is forty-one. Find the number.

Solution:

38

Exercise:

Problem:

Twice the difference of a number and ten is fifty-four. Find the number.

Exercise:

Problem:

One number is nine less than another. Their sum is twenty-seven. Find the numbers.

Solution:

18, 9

Exercise:

Problem:

The sum of two consecutive integers is -135. Find the numbers.

Solve Money Applications

Solve Coin Word Problems

In the following exercises, solve each coin word problem.

Exercise:

Problem:

Francie has $\$4.35$ in dimes and quarters. The number of dimes is 5 more than the number of quarters. How many of each coin does she have?

Solution:

16 dimes, 11 quarters

Exercise:

Problem:

Scott has $\$0.39$ in pennies and nickels. The number of pennies is 8 times the number of nickels. How many of each coin does he have?

Exercise:

Problem:

Paulette has $140 in $5 and $10 bills. The number of $10 bills is one less than twice the number of $5 bills. How many of each does she have?

Solution:

6 of $5 bills, 11 of $10 bills

Exercise:

Problem:

Lenny has $\$3.69$ in pennies, dimes, and quarters. The number of pennies is 3 more than the number of dimes. The number of quarters is twice the number of dimes. How many of each coin does he have?

Solve Ticket and Stamp Word Problems

In the following exercises, solve each ticket or stamp word problem.

Exercise:

Problem:

A church luncheon made $842. Adult tickets cost $10 each and children’s tickets cost $6 each. The number of children was 12 more than twice the number of adults. How many of each ticket were sold?

Solution:

35 adults, 82 children

Exercise:

Problem:

Tickets for a basketball game cost $2 for students and $5 for adults. The number of students was 3 less than 10 times the number of adults. The total amount of money from ticket sales was $619. How many of each ticket were sold?

Exercise:

Problem:

Ana spent $4.06 buying stamps. The number of $0.41 stamps she bought was 5 more than the number of $0.26 stamps. How many of each did she buy?

Solution:

3 of 26 -cent stamps, 8 of 41 -cent stamps

Exercise:

Problem:

Yumi spent $34.15 buying stamps. The number of $0.56 stamps she bought was 10 less than 4 times the number of $0.41 stamps. How many of each did she buy?

Use Properties of Angles, Triangles, and the Pythagorean Theorem

Use Properties of Angles

In the following exercises, solve using properties of angles.

Exercise:

Problem: What is the supplement of a 48° angle?

Solution:

132°

Exercise:

Problem: What is the complement of a 61° angle?

Exercise:

Problem:

Two angles are complementary. The smaller angle is 24° less than the larger angle. Find the measures of both angles.

Solution:

33°, 57°

Exercise:

Problem:

Two angles are supplementary. The larger angle is 45° more than the smaller angle. Find the measures of both angles.

Use Properties of Triangles

In the following exercises, solve using properties of triangles.

Exercise:

Problem:

The measures of two angles of a triangle are 22 and 85 degrees. Find the measure of the third angle.

Solution:

73°

Exercise:

Problem:

One angle of a right triangle measures 41.5 degrees. What is the measure of the other small angle?

Exercise:

Problem:

One angle of a triangle is 30° more than the smallest angle. The largest angle is the sum of the other angles. Find the measures of all three angles.

Solution:

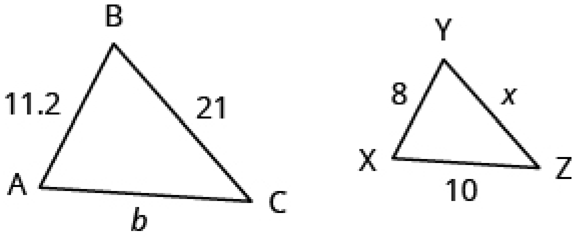
30°, 60°, 90°

Exercise:

Problem:

One angle of a triangle is twice the measure of the smallest angle. The third angle is 60° more than the measure of the smallest angle. Find the measures of all three angles.

In the following exercises, ΔABC is similar to ΔXYZ. Find the length of the indicated side.



Exercise:

Problem: side x

Solution:

15

Exercise:

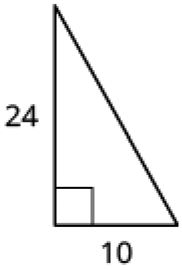
Problem: side b

Use the Pythagorean Theorem

In the following exercises, use the Pythagorean Theorem to find the length of the missing side. Round to the nearest tenth, if necessary.

Exercise:

Problem:

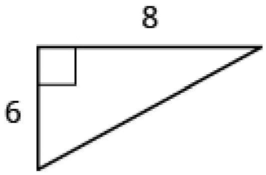


Solution:

26

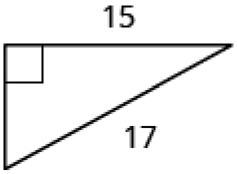
Exercise:

Problem:



Exercise:

Problem:

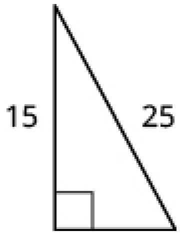


Solution:

8

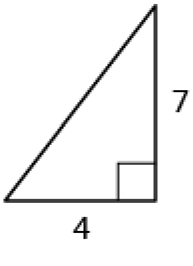
Exercise:

Problem:



Exercise:

Problem:

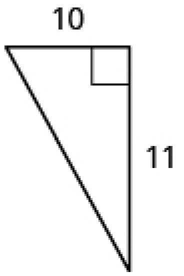


Solution:

8.1

Exercise:

Problem:

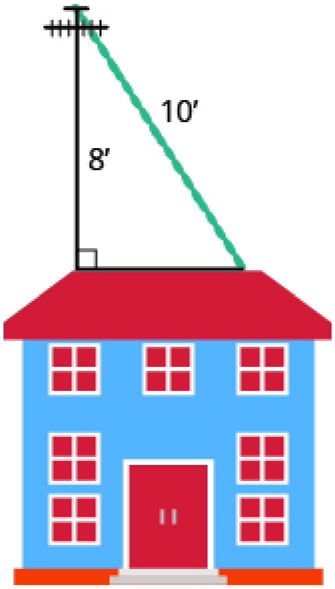


In the following exercises, solve. Approximate to the nearest tenth, if necessary.

Exercise:

Problem:

Sergio needs to attach a wire to hold the antenna to the roof of his house, as shown in the figure. The antenna is 8 feet tall and Sergio has 10 feet of wire. How far from the base of the antenna can he attach the wire?



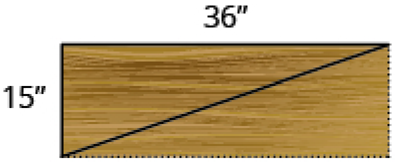
Solution:

6 feet

Exercise:

Problem:

Seong is building shelving in his garage. The shelves are 36 inches wide and 15 inches tall. He wants to put a diagonal brace across the back to stabilize the shelves, as shown. How long should the brace be?



Use Properties of Rectangles, Triangles, and Trapezoids

Understand Linear, Square, Cubic Measure

In the following exercises, would you measure each item using linear, square, or cubic measure?

Exercise:

Problem: amount of sand in a sandbag

Solution:

cubic

Exercise:

Problem: height of a tree

Exercise:

Problem: size of a patio

Solution:

square

Exercise:

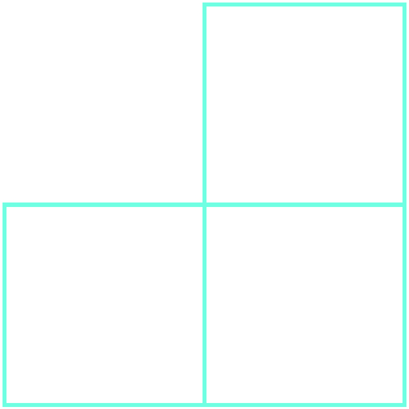
Problem: length of a highway

In the following exercises, find

ⓐ the perimeter
ⓑ the area of each figure

Exercise:

Problem:

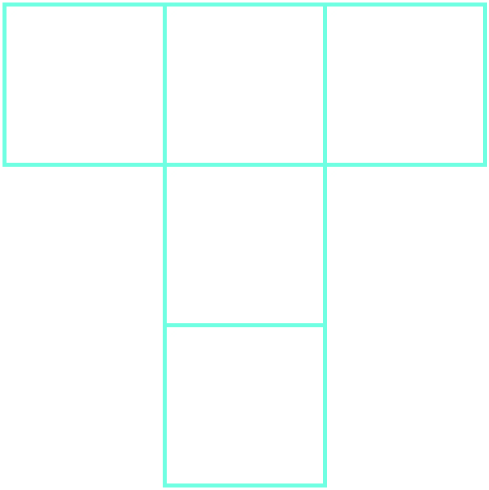


Solution:

ⓐ 8 units
ⓑ 3 sq. units

Exercise:

Problem:



Use Properties of Rectangles

In the following exercises, find the ⓐ perimeter ⓑ area of each rectangle
Exercise:

Problem:

The length of a rectangle is 42 meters and the width is 28 meters.

Solution:

ⓐ 140 m
ⓑ 1176 sq. m

Exercise:

Problem: The length of a rectangle is 36 feet and the width is 19 feet.

Exercise:

Problem:

A sidewalk in front of Kathy’s house is in the shape of a rectangle 4 feet wide by 45 feet long.

Solution:

ⓐ 98 ft.
ⓑ 180 sq. ft.

Exercise:

Problem: A rectangular room is 16 feet wide by 12 feet long.

In the following exercises, solve.

Exercise:

Problem:

Find the length of a rectangle with perimeter of 220 centimeters and width of 85 centimeters.

Solution:

25 cm

Exercise:

Problem:

Find the width of a rectangle with perimeter 39 and length 11.

Exercise:

Problem:

The area of a rectangle is 2356 square meters. The length is 38 meters. What is the width?

Solution:

62 m

Exercise:

Problem:

The width of a rectangle is 45 centimeters. The area is 2700 square centimeters. What is the length?

Exercise:

Problem:

The length of a rectangle is 12 centimeters more than the width. The perimeter is 74 centimeters. Find the length and the width.

Solution:

24.5 in., 12.5 in.

Exercise:

Problem:

The width of a rectangle is 3 more than twice the length. The perimeter is 96 inches. Find the length and the width.

Use Properties of Triangles

In the following exercises, solve using the properties of triangles.

Exercise:

Problem:

Find the area of a triangle with base 18 inches and height 15 inches.

Solution:

135 sq. in.

Exercise:

Problem:

Find the area of a triangle with base 33 centimeters and height 21 centimeters.

Exercise:

Problem:

A triangular road sign has base 30 inches and height 40 inches. What is its area?

Solution:

600 sq. in.

Exercise:

Problem:

If a triangular courtyard has sides 9 feet and 12 feet and the perimeter is 32 feet, how long is the third side?

Exercise:

Problem:

A tile in the shape of an isosceles triangle has a base of 6 inches. If the perimeter is 20 inches, find the length of each of the other sides.

Solution:

7 in., 7 in.

Exercise:

Problem:

Find the length of each side of an equilateral triangle with perimeter of 81 yards.

Exercise:

Problem:

The perimeter of a triangle is 59 feet. One side of the triangle is 3 feet longer than the shortest side. The third side is 5 feet longer than the shortest side. Find the length of each side.

Solution:

17 ft., 20 ft., 22 ft.

Exercise:

Problem:

One side of a triangle is three times the smallest side. The third side is 9 feet more than the shortest side. The perimeter is 39 feet. Find the lengths of all three sides.

Use Properties of Trapezoids

In the following exercises, solve using the properties of trapezoids.

Exercise:

Problem:

The height of a trapezoid is 8 feet and the bases are 11 and 14 feet. What is the area?

Solution:

100 sq. ft.

Exercise:

Problem:

The height of a trapezoid is 5 yards and the bases are 7 and 10 yards. What is the area?

Exercise:

Problem:

Find the area of the trapezoid with height 25 meters and bases 32.5 and 21.5 meters.

Solution:

675 sq. m

Exercise:

Problem:

A flag is shaped like a trapezoid with height 62 centimeters and the bases are 91.5 and 78.1 centimeters. What is the area of the flag?

Solve Geometry Applications: Circles and Irregular Figures

Use Properties of Circles

In the following exercises, solve using the properties of circles. Round answers to the nearest hundredth.

Exercise:

Problem: A circular mosaic has radius 3 meters. Find the

ⓐ circumference
ⓑ area of the mosaic

Solution:

ⓐ 18.84 m
ⓑ 28.26 sq. m

Exercise:

Problem: A circular fountain has radius 8 feet. Find the

ⓐ circumference
ⓑ area of the fountain

Exercise:

Problem:

Find the diameter of a circle with circumference 150.72 inches.

Solution:

48 in.

Exercise:

Problem:

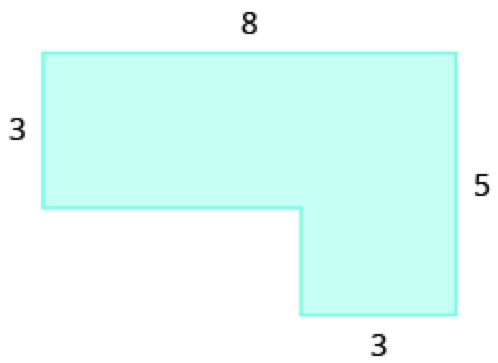
Find the radius of a circle with circumference 345.4 centimeters

Find the Area of Irregular Figures

In the following exercises, find the area of each shaded region.

Exercise:

Problem:

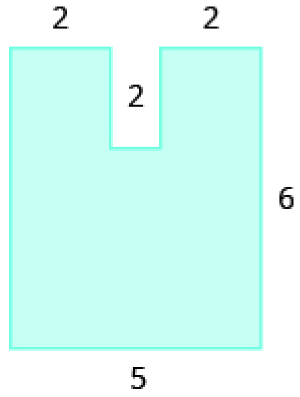


Solution:

30 sq. units

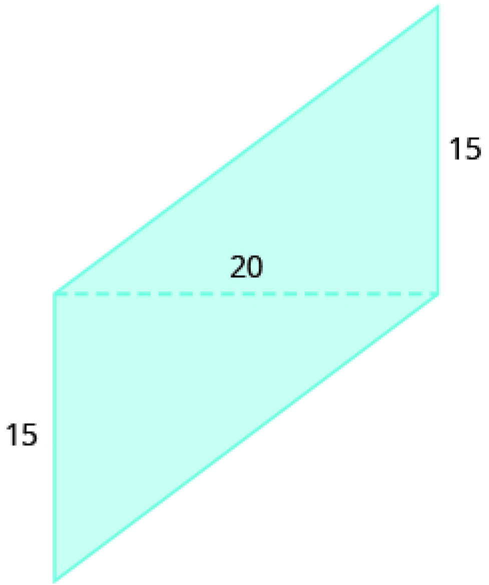
Exercise:

Problem:



Exercise:

Problem:

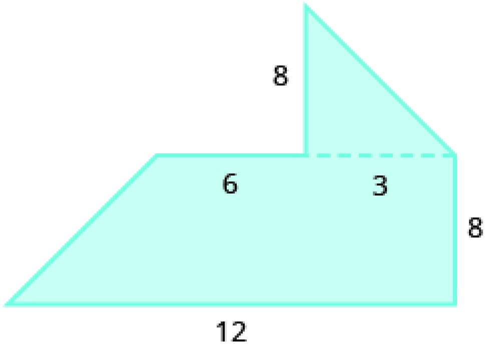


Solution:

300 sq. units

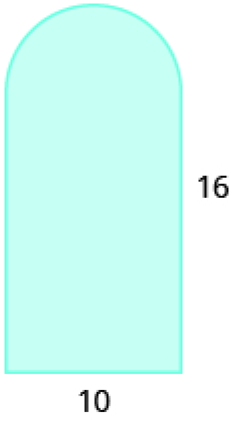
Exercise:

Problem:



Exercise:

Problem:

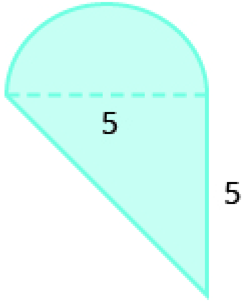


Solution:

199.25 sq. units

Exercise:

Problem:



[Solve Geometry Applications: Volume and Surface Area]()

Find Volume and Surface Area of Rectangular Solids

In the following exercises, find the

ⓐ volume
ⓑ surface area of the rectangular solid

Exercise:

Problem:

a rectangular solid with length 14 centimeters, width 4.5 centimeters, and height 10 centimeters

---

Solution:

ⓐ 630 cu. cm
ⓑ 496 sq. cm

Exercise:

Problem: a cube with sides that are 3 feet long

Exercise:

Problem: a cube of tofu with sides 2.5 inches

Solution:

ⓐ 15.625 cu. in.
ⓑ 37.5 sq. in.

Exercise:

Problem:

a rectangular carton with length 32 inches, width 18 inches, and height 10 inches

Find Volume and Surface Area of Spheres

In the following exercises, find the

ⓐ volume
ⓑ surface area of the sphere.

Exercise:

Problem: a sphere with radius 4 yards

Solution:

ⓐ 267.95 cu. yd.
ⓑ 200.96 sq. yd.

Exercise:

Problem: a sphere with radius 12 meters

Exercise:

Problem: a baseball with radius 1.45 inches

Solution:

ⓐ 12.76 cu. in.
ⓑ 26.41 sq. in.

Exercise:

Problem: a soccer ball with radius 22 centimeters

Find Volume and Surface Area of Cylinders

In the following exercises, find the

ⓐ volume
ⓑ surface area of the cylinder

Exercise:

Problem: a cylinder with radius 2 yards and height 6 yards

Solution:

ⓐ 75.36 cu. yd.
ⓑ 100.48 sq. yd.

Exercise:

Problem: a cylinder with diameter 18 inches and height 40 inches

Exercise:

Problem:

a juice can with diameter 8 centimeters and height 15 centimeters

---

Solution:

ⓐ 753.6 cu. cm
ⓑ 477.28 sq. cm

Exercise:

Problem:

a cylindrical pylon with diameter 0.8 feet and height 2.5 feet

Find Volume of Cones

In the following exercises, find the volume of the cone.

Exercise:

Problem: a cone with height 5 meters and radius 1 meter

Solution:

5.233 cu. m

Exercise:

Problem: a cone with height 24 feet and radius 8 feet

Exercise:

Problem:

a cone-shaped water cup with diameter 2.6 inches and height 2.6 inches

---

Solution:

4.599 cu. in.

Exercise:

Problem:

a cone-shaped pile of gravel with diameter 6 yards and height 5 yards

Solve a Formula for a Specific Variable

Use the Distance, Rate, and Time Formula

In the following exercises, solve using the formula for distance, rate, and time.

Exercise:

Problem:

A plane flew 4 hours at 380 miles per hour. What distance was covered?

Solution:

1520 miles

Exercise:

Problem:

Gus rode his bike for $1\frac{1}{2}$ hours at 8 miles per hour. How far did he ride?

Exercise:

Problem:

Jack is driving from Bangor to Portland at a rate of 68 miles per hour. The distance is 107 miles. To the nearest tenth of an hour, how long will the trip take?

Solution:

1.6 hours

Exercise:

Problem:

Jasmine took the bus from Pittsburgh to Philadelphia. The distance is 305 miles and the trip took 5 hours. What was the speed of the bus?

Solve a Formula for a Specific Variable

In the following exercises, use the formula $d = rt$.

Exercise:

Problem: Solve for t:

ⓐ when $d = 403$ and $r = 65$
ⓑ in general

Solution:

ⓐ $t = 6.2$
ⓑ $t = \frac{d}{r}$

Exercise:

Problem: Solve for r:

ⓐ when $d = 750$ and $t = 15$
ⓑ in general

In the following exercises, use the formula $A = \frac{1}{2}bh$.

Exercise:

Problem: Solve for b:

ⓐ when $A = 416$ and $h = 32$
ⓑ in general

Solution:

ⓐ $b = 26$
ⓑ $b = \frac{2A}{h}$

Exercise:

Problem: Solve for h:

ⓐ when $A = 48$ and $b = 8$
ⓑ in general

In the following exercises, use the formula $I = Prt$.

Exercise:

Problem: Solve for the principal, P, for:

ⓐ $I = \$720$, $r = 4\%$, $t = 3$ years
ⓑ in general

Solution:

ⓐ $P = \$6000$
ⓑ $P = \frac{I}{(r \cdot t)}$

Exercise:

Problem: Solve for the time, t for:

ⓐ $I = \$3630$, $P = \$11{,}000$, $r = 5.5\%$
ⓑ in general

In the following exercises, solve.
Exercise:

Problem: Solve the formula $6x + 5y = 20$ for y:

ⓐ when $x = 0$
ⓑ in general

Solution:

ⓐ $y = 4$
ⓑ $y = \frac{20 - 6x}{5}$

Exercise:

Problem: Solve the formula $2x + y = 15$ for y:

ⓐ when $x = -5$

ⓑ in general

Exercise:

Problem: Solve $a + b = 90$ for a.

Solution:

$a = 90 - b$

Exercise:

Problem: Solve $180 = a + b + c$ for a.

Exercise:

Problem: Solve the formula $4x + y = 17$ for y.

Solution:

$y = 17 - 4x$

Exercise:

Problem: Solve the formula $-3x + y = -6$ for y.

Exercise:

Problem: Solve the formula $P = 2L + 2W$ for W.

Solution:

$W = \frac{P-2L}{2}$

Exercise:

Problem: Solve the formula $V = LWH$ for H.

Exercise:

Problem:

Describe how you have used two topics from this chapter in your life outside of math class during the past month.

Chapter Practice Test

Exercise:

Problem:

Four-fifths of the people on a hike are children. If there are 12 children, what is the total number of people on the hike?

Exercise:

Problem:

The sum of 13 and twice a number is -19. Find the number.

Solution:

−16

Exercise:

Problem:

One number is 3 less than another number. Their sum is 65. Find the numbers.

Exercise:

Problem:

Bonita has $\$2.95$ in dimes and quarters in her pocket. If she has 5 more dimes than quarters, how many of each coin does she have?

Solution:

7 quarters, 12 dimes

Exercise:

Problem:

At a concert, $\$1600$ in tickets were sold. Adult tickets were $\$9$ each and children's tickets were $\$4$ each. If the number of adult tickets was 30 fewer than twice the number of children's tickets, how many of each kind were sold?

Exercise:

Problem: Find the complement of a 52° angle.

Solution:

38°

Exercise:

Problem:

The measure of one angle of a triangle is twice the measure of the smallest angle. The measure of the third angle is 14 more than the measure of the smallest angle. Find the measures of all three angles.

Exercise:

Problem:

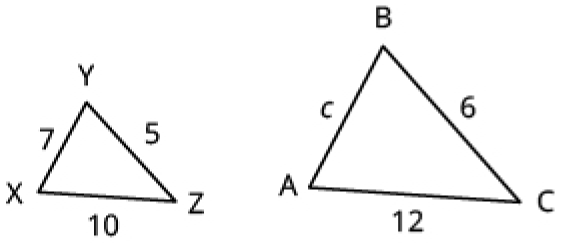
The perimeter of an equilateral triangle is 145 feet. Find the length of each side.

Solution:

48.3

Exercise:

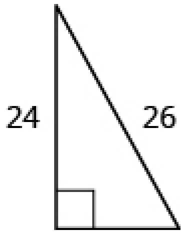
Problem: ΔABC is similar to ΔXYZ. Find the length of side c.


Y
7
5
X
Z
10
B
c
6
A
C
12

Exercise:

Problem:

Find the length of the missing side. Round to the nearest tenth, if necessary.



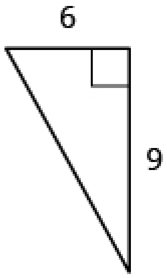
Solution:

10

Exercise:

Problem:

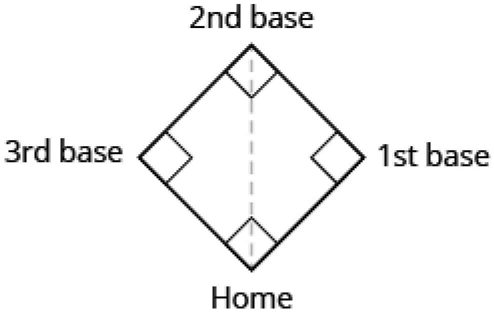
Find the length of the missing side. Round to the nearest tenth, if necessary.



Exercise:

Problem:

A baseball diamond is shaped like a square with sides 90 feet long. How far is it from home plate to second base, as shown?



Solution:

127.3 ft

Exercise:

Problem:

The length of a rectangle is 2 feet more than five times the width. The perimeter is 40 feet. Find the dimensions of the rectangle.

Exercise:

Problem:

A triangular poster has base 80 centimeters and height 55 centimeters. Find the area of the poster.

Solution:

2200 square centimeters

Exercise:

Problem:

A trapezoid has height 14 inches and bases 20 inches and 23 inches. Find the area of the trapezoid.

Exercise:

Problem:

A circular pool has diameter 90 inches. What is its circumference? Round to the nearest *tenth.*

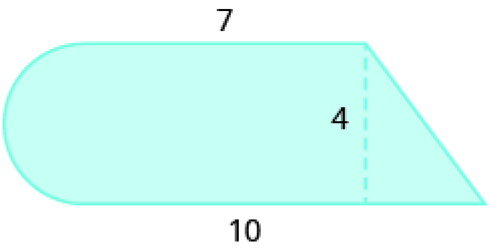
Solution:

282.6 inches

Exercise:

Problem:

Find the area of the shaded region. Round to the nearest tenth.



Exercise:

Problem:

Find the volume of a rectangular room with width 12 feet, length 15 feet, and height 8 feet.

Solution:

1440

Exercise:

Problem:

A coffee can is shaped like a cylinder with height 7 inches and radius 5 inches. Find (a) the surface area and (b) the volume of the can. Round to the nearest tenth.

Exercise:

Problem:

A traffic cone has height 75 centimeters. The radius of the base is 20 centimeters. Find the volume of the cone. Round to the nearest tenth.

Solution:

31,400 cubic inches

Exercise:

Problem:

Leon drove from his house in Cincinnati to his sister's house in Cleveland. He drove at a uniform rate of 63 miles per hour and the trip took 4 hours. What was the distance?

Exercise:

Problem:

The Catalina Express takes $1\frac{1}{2}$ hours to travel from Long Beach to Catalina Island, a distance of 22 miles. To the nearest tenth, what is the speed of the boat?

Solution:

14.7 miles per hour

Exercise:

Problem: Use the formula $I = Prt$ to solve for the principal, P, for:

ⓐ $I = \$1380,\ r = 5\%,\ t = 3$ years
ⓑ in general

Exercise:

Problem: Solve the formula $A = \frac{1}{2}bh$ for h:

ⓐ when $A = 1716$ and $b = 66$
ⓑ in general

Solution:

ⓐ $\text{height} = 52$
ⓑ $h = \frac{2A}{b}$

Exercise:

Problem: Solve $x + 5y = 14$ for y.

Solve Linear Inequalities
By the end of this section, you will be able to:

- Graph inequalities on the number line
- Solve linear inequalities
- Translate words to an inequality and solve
- Solve applications with linear inequalities

Note:
Before you get started, take this readiness quiz.

1. Translate from algebra to English: $15 > x$.
   If you missed this problem, review [link].
2. Translate to an algebraic expression: 15 is less than *x*.
   If you missed this problem, review [link].

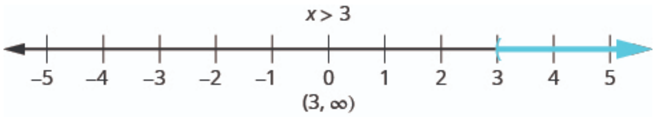
Graph Inequalities on the Number Line

What number would make the inequality $x > 3$ true? Are you thinking, "*x* could be four"? That's correct, but *x* could be 6, too, or 37, or even 3.001. Any number greater than three is a solution to the inequality $x > 3$.

We show all the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of three, to show that all numbers greater than three are solutions. Because the number three itself is not a solution, we put an open parenthesis at three.

We can also represent inequalities using **interval notation**. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as "**infinity**." It is not an actual number.

[link] shows both the number line and the interval notation.

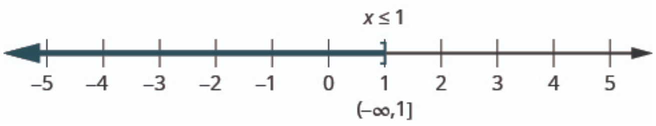


The inequality $x > 3$ is graphed on this number line and written in interval notation.

We use the left parenthesis symbol, (, to show that the endpoint of the inequality is not included. The left bracket symbol, [, shows that the endpoint is included.

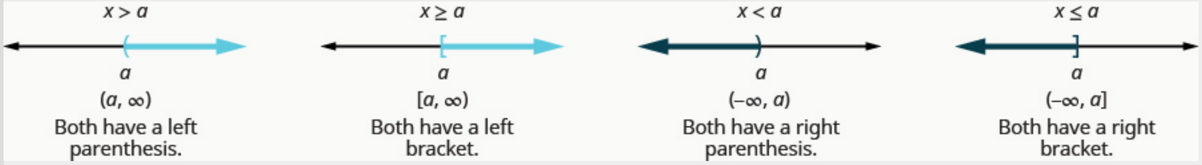
The inequality $x \leq 1$ means all numbers less than or equal to one. Here we need to show that one is a solution, too. We do that by putting a bracket at $x = 1$. We then shade in all the numbers to the left of one, to show that all numbers less than one are solutions. See [link].

There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as "negative infinity." [link] shows both the number line and interval notation.


$x \leq 1$
–5
–4
–3
–2
–1
0
1
2
3
4
5
(–∞,1]

The inequality $x \leq 1$ is graphed on this number line and written in interval notation.

Note:
Inequalities, Number Lines, and Interval Notation


$x > a$
a
(a, ∞)
Both have a left parenthesis.
$x \geq a$
a
$[a, \infty)$
Both have a left bracket.
$x < a$
a
$(-\infty, a)$
Both have a right parenthesis.
$x \leq a$
a
$(-\infty, a]$
Both have a right bracket.

The notation for inequalities on a number line and in interval notation use the same symbols to express the endpoints of intervals.

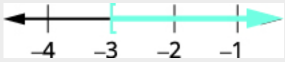
Example:
Exercise:

Problem: Graph each inequality on the number line and write in interval notation.

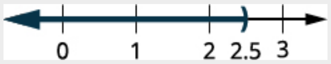
ⓐ $x \geq -3$ ⓑ $x < 2.5$ ⓒ $x \leq -\frac{3}{5}$

Solution:

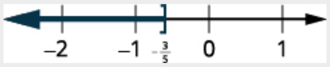
ⓐ

	$x \geq -3$
Shade to the right of -3, and put a bracket at -3.	
Write in interval notation.	$[-3, \infty)$

ⓑ

	$x < 2.5$
Shade to the left of 2.5 and put a parenthesis at 2.5.	
Write in interval notation.	$(-\infty, 2.5)$

ⓒ

	$x \leq -\frac{3}{5}$
Shade to the left of $-\frac{3}{5}$, and put a bracket at $-\frac{3}{5}$.	
Write in interval notation.	$\left(-\infty, -\frac{3}{5}\right]$

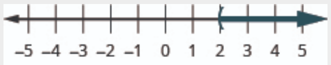
Note:
Exercise:

Problem:

Graph each inequality on the number line and write in interval notation: ⓐ $x > 2$ ⓑ $x \leq -1.5$ ⓒ $x \geq \frac{3}{4}$.

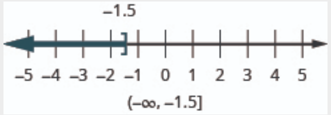
Solution:

ⓐ

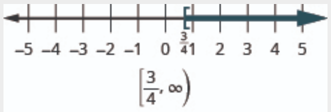


$(2, \infty)$

ⓑ


–1.5
–5 –4 –3 –2 –1 0 1 2 3 4 5
(–∞, –1.5]

ⓒ


–5 –4 –3 –2 –1 0 3/4 1 2 3 4 5
[3/4, ∞)

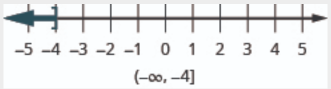
Note:
Exercise:

Problem:

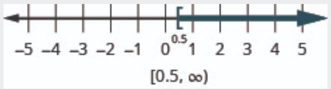
Graph each inequality on the number line and write in interval notation: ⓐ $x \le -4$ ⓑ$x \ge 0.5$ ⓒ $x < -\frac{2}{3}$.

Solution:

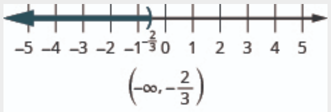
ⓐ


–5 –4 –3 –2 –1 0 1 2 3 4 5
(–∞, –4]

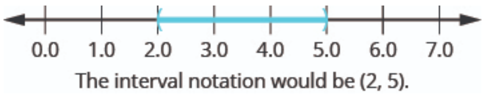
ⓑ


–5 –4 –3 –2 –1 0 0.5 1 2 3 4 5
[0.5, ∞)

ⓒ


–5 –4 –3 –2 –1 –2/3 0 1 2 3 4 5
(–∞, –2/3)

What numbers are greater than two but less than five? Are you thinking say, $2.5, 3, 3\frac{2}{3}, 4, 4, 99$? We can represent all the numbers between two and five with the inequality $2 < x < 5$. We can show $2 < x < 5$ on the number line by shading all the numbers between two and five. Again, we use the parentheses to show the numbers two and five are not included. See [link].


0.0 1.0 2.0 3.0 4.0 5.0 6.0 7.0

The interval notation would be (2, 5).

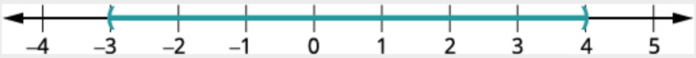
Example:
Exercise:

Problem: Graph each inequality on the number line and write in interval notation.

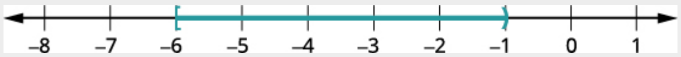
ⓐ $-3 < x < 4$ ⓑ $-6 \leq x < -1$ ⓒ $0 \leq x \leq 2.5$

Solution:

ⓐ

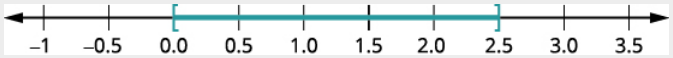
		$-3 < x < 4$
Shade between -3 and 4. Put a parentheses at -3 and 4.		
Write in interval notation.		$(-3, 4)$

ⓑ

			$-6 \leq x < -1$
Shade between -6 and -1. Put a bracket at -6, and a parenthesis at -1.			
Write in interval notation.			$[-6, 1)$

ⓒ

			$0 \leq x \leq 2.5$

Shade between 0 and 2.5. Put a bracket at 0 and at 2.5.			 –1 –0.5 0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5
Write in interval notation.			[0, 2.5]

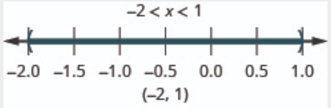
Note:
Exercise:

Problem: Graph each inequality on the number line and write in interval notation:

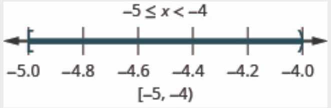
ⓐ $-2 < x < 1$ ⓑ $-5 \leq x < -4$ ⓒ $1 \leq x \leq 4.25$

Solution:

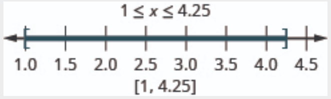
ⓐ


–2 < x < 1
–2.0
–1.5
–1.0
–0.5
0.0
0.5
1.0
(–2, 1)

ⓑ


–5 ≤ x < –4
–5.0
–4.8
–4.6
–4.4
–4.2
–4.0
[–5, –4)

ⓒ


1 ≤ x ≤ 4.25
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
[1, 4.25]

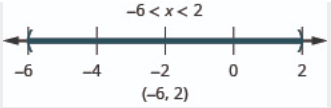
Note:
Exercise:

Problem: Graph each inequality on the number line and write in interval notation:

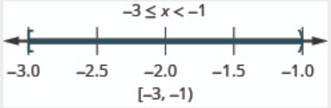
ⓐ $-6 < x < 2$ ⓑ $-3 \leq x < -1$ ⓒ $2.5 \leq x \leq 6$

Solution:

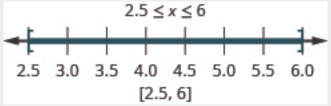
ⓐ



ⓑ



ⓒ



Solve Linear Inequalities

A linear inequality is much like a linear equation—but the equal sign is replaced with an inequality sign. A **linear inequality** is an inequality in one variable that can be written in one of the forms, $ax+b<c$,$ax+b\leq c$, $ax+b>c$, or $ax+b\geq c$.

Note:
Linear Inequality
A linear inequality is an inequality in one variable that can be written in one of the following forms where a, b, and c are real numbers and $a\neq 0$:
Equation:

$$ax+b<c, \qquad ax+b\leq c, \qquad ax+b>c, \qquad ax+b\geq c.$$

When we solved linear equations, we were able to use the properties of equality to add, subtract, multiply, or divide both sides and still keep the equality. Similar properties hold true for inequalities.

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality. For example:

$$\begin{array}{rcl} -4<2 & \qquad & -4<2 \\ -4-5<2-5 & \qquad & -4+7<2+7 \\ -9<-3 \text{ True} & \qquad & 3<9 \text{ True} \end{array}$$

Notice that the inequality sign stayed the same.

This leads us to the Addition and Subtraction Properties of Inequality.

Note:
Addition and Subtraction Property of Inequality

For any numbers a, b, and c, if $a < b$, then

Equation:

$$\begin{array}{ll} a+c<b+c & a-c<b-c \\ a+c>b+c & a-c>b-c \end{array}$$

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality.

What happens to an inequality when we divide or multiply both sides by a constant?

Let's first multiply and divide both sides by a positive number.

$$\begin{array}{cc} 10<15 & 10<15 \\ 10(5)<15(5) & \frac{10}{5}<\frac{15}{5} \\ 50<75 \text{ True} & 2<3 \text{ True} \end{array}$$

The inequality signs stayed the same.

Does the inequality stay the same when we divide or multiply by a negative number?

$$\begin{array}{cc} 10<15 & 10<15 \\ 10(-5)\ ?\ 15(-5) & \frac{10}{-5}\ ?\ \frac{15}{-5} \\ -50\ ?\ -75 & -2\ ?\ -3 \\ -50>-75 & -2>-3 \end{array}$$

Notice that when we filled in the inequality signs, the inequality signs reversed their direction.

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses.

This gives us the Multiplication and Division Property of Inequality.

Note:

Multiplication and Division Property of Inequality

For any numbers a, b, and c,

Equation:

$$\begin{array}{l} \text{multiply or divide by a positive} \\ \qquad \text{if } a<b \text{ and } c>0, \text{then } ac<bc \text{ and } \frac{a}{c}<\frac{b}{c}. \\ \qquad \text{if } a>b \text{ and } c>0, \text{then } ac>bc \text{ and } \frac{a}{c}>\frac{b}{c}. \\ \text{multiply or divide by a negative} \\ \qquad \text{if } a<b \text{ and } c<0, \text{then } ac>bc \text{ and } \frac{a}{c}>\frac{b}{c}. \\ \qquad \text{if } a>b \text{ and } c<0, \text{then } ac<bc \text{ and } \frac{a}{c}<\frac{b}{c}. \end{array}$$

When we divide or multiply an inequality by a:

- positive number, the inequality stays the same.
- negative number, the inequality reverses.

Sometimes when solving an inequality, as in the next example, the variable ends upon the right. We can rewrite the inequality in reverse to get the variable to the left.

Equation:

$$x > a \text{ has the same meaning as } a < x$$

Think about it as “If Xander is taller than Andy, then Andy is shorter than Xander.”

Example:

Exercise:

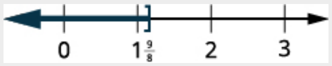
Problem:

Solve each inequality. Graph the solution on the number line, and write the solution in interval notation.

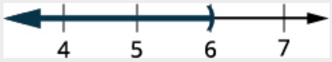
ⓐ $x-\frac{3}{8}\le\frac{3}{4}$ ⓑ $9y<54$ ⓒ $-15<\frac{3}{5}z$

Solution:

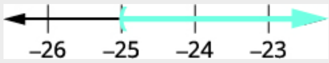
ⓐ

	$x-\frac{3}{8}\le\frac{3}{4}$
Add $\frac{3}{8}$ to both sides of the inequality.	$x-\frac{3}{8}+\frac{3}{8}\le\frac{3}{4}+\frac{3}{8}$
Simplify.	$x\le\frac{9}{8}$
Graph the solution on the number line.	
Write the solution in interval notation.	$\left(-\infty,\frac{9}{8}\right]$

ⓑ

	$9y < 54$
Divide both sides of the inequality by 9; since 9 is positive, the inequality stays the same.	$\frac{9y}{9} < \frac{54}{9}$
Simplify.	$y < 6$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, 6)$

ⓒ

	$-15 < \frac{3}{5}z$
Multiply both sides of the inequality by $\frac{5}{3}$. Since $\frac{5}{3}$ is positive, the inequality stays the same.	$\frac{5}{3}(-15) < \frac{5}{3}\left(\frac{3}{5}z\right)$
Simplify.	$-25 < z$
Rewrite with the variable on the left.	$z > -25$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-25, \infty)$

Note:
Exercise:

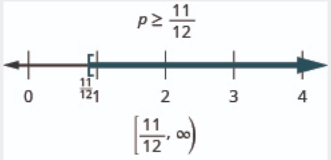
Problem:

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

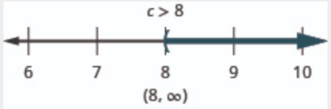
ⓐ $p-\frac{3}{4}\geq\frac{1}{6}$ ⓑ $9c>72$ ⓒ $24\leq\frac{3}{8}m$

Solution:

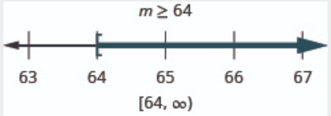
ⓐ


$p\geq\frac{11}{12}$
0
$\frac{11}{12}$
1
2
3
4
$\left[\frac{11}{12},\infty\right)$

ⓑ


$c>8$
6
7
8
9
10
$(8,\infty)$

ⓒ


$m\geq 64$
63
64
65
66
67
$[64,\infty)$

Note:
Exercise:

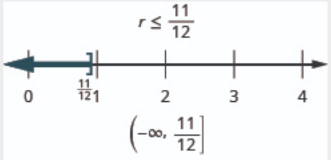
Problem:

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

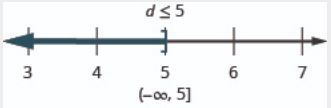
ⓐ $r-\frac{1}{3}\leq\frac{7}{12}$ ⓑ $12d\leq 60$ ⓒ $-24<\frac{4}{3}n$

Solution:

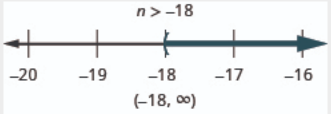
ⓐ


$r\leq\frac{11}{12}$
0
$\frac{11}{12}$
1
2
3
4
$\left(-\infty,\frac{11}{12}\right]$

ⓑ


$d\leq 5$
3
4
5
6
7
$(-\infty,5]$

ⓒ



Be careful when you multiply or divide by a negative number—remember to reverse the inequality sign.

Example:
Exercise:

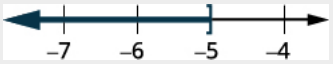
Problem:

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

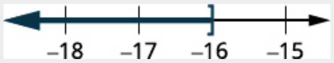
ⓐ $-13m \geq 65$ ⓑ $\frac{n}{-2} \geq 8$

Solution:

ⓐ

	$-13m \geq 65$
Divide both sides of the inequality by -13. Since -13 is a negative, the inequality reverses.	$\frac{-13m}{-13} \leq \frac{65}{-13}$
Simplify.	$m \leq -5$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -5]$

ⓑ

	$\frac{n}{-2} \geq 8$
Multiply both sides of the inequality by -2. Since -2 is a negative, the inequality reverses.	$-2\left(\frac{n}{-2}\right) \leq -2(8)$
Simplify.	$n \leq -16$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -16]$

Note:
Exercise:

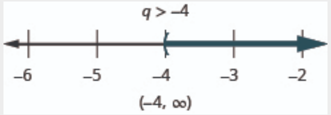
Problem:

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

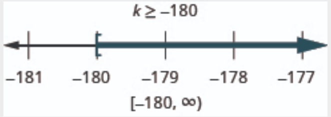
ⓐ $-8q < 32$ ⓑ $\frac{k}{-12} \leq 15$.

Solution:

ⓐ



ⓑ



Note:
Exercise:

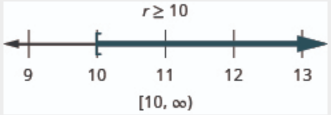
Problem:

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

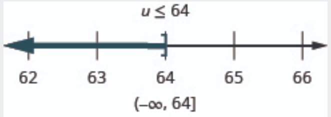
ⓐ $-7r \le -70$ ⓑ $\frac{u}{-4} \ge -16$.

Solution:

ⓐ



ⓑ



Most inequalities will take more than one step to solve. We follow the same steps we used in the general strategy for solving linear equations, but make sure to pay close attention when we multiply or divide to isolate the variable.

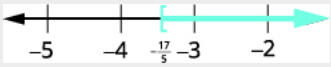
Example:
Exercise:

Problem:

Solve the inequality $6y \le 11y + 17$, graph the solution on the number line, and write the solution in interval notation.

Solution:

	$6y \le 11y + 17$
Subtract $11y$ from both sides to collect the variables on the left.	$6y - 11y \le 11y - 11y + 17$
Simplify.	$-5y \le 17$
Divide both sides of the inequality by -5, and reverse the inequality.	$\frac{-5y}{-5} \ge \frac{17}{-5}$
Simplify.	

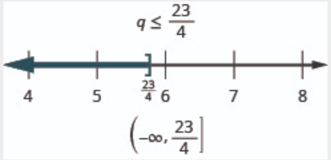
	$y \geq \frac{17}{-5}$
Graph the solution on the number line.	
Write the solution in interval notation.	$\left[-\frac{17}{5}, \infty\right)$

Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation: $3q \geq 7q - 23$.

Solution:



$q \leq \frac{23}{4}$

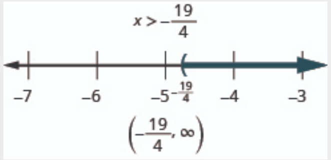
$\left(-\infty, \frac{23}{4}\right]$

Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation: $6x < 10x + 19$.

Solution:



$x > -\frac{19}{4}$

$\left(-\frac{19}{4}, \infty\right)$

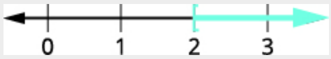
When solving inequalities, it is usually easiest to collect the variables on the side where the coefficient of the variable is largest. This eliminates negative coefficients and so we don't have to multiply or divide by a negative—which means we don't have to remember to reverse the inequality sign.

Example:
Exercise:

Problem:

Solve the inequality $8p+3\left(p-12\right)>7p-28$, graph the solution on the number line, and write the solution in interval notation.

Solution:

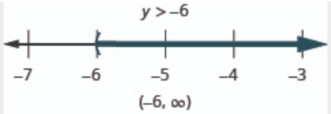
	$8p+3(p-12)>7p-28$
Simplify each side as much as possible.	
Distribute.	$8p+3p-36>7p-28$
Combine like terms.	$11p-36>7p-28$
Subtract $7p$ from both sides to collect the variables on the left, since $11>7$.	$11p-36-7p>7p-28-7p$
Simplify.	$4p-36>-28$
Add 36 to both sides to collect the constants on the right.	$4p-36+36>-28+36$
Simplify.	$4p>8$
Divide both sides of the inequality by 4; the inequality stays the same.	$\frac{4p}{4}>\frac{8}{4}$
Simplify.	$p>2$
Graph the solution on the number line.	
Write the solution in interval notation.	$(2,\infty)$

Note:
Exercise:

Problem:

Solve the inequality $9y+2\left(y+6\right)>5y-24$, graph the solution on the number line, and write the solution in interval notation.

Solution:

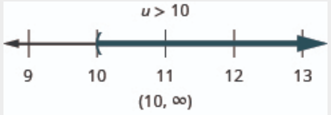


Note:
Exercise:

Problem:

Solve the inequality $6u+8(u-1)>10u+32$, graph the solution on the number line, and write the solution in interval notation.

Solution:



Just like some equations are identities and some are contradictions, inequalities may be identities or contradictions, too. We recognize these forms when we are left with only constants as we solve the inequality. If the result is a true statement, we have an identity. If the result is a false statement, we have a contradiction.

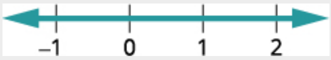
Example:
Exercise:

Problem:

Solve the inequality $8x-2\left(5-x\right)<4\left(x+9\right)+6x$, graph the solution on the number line, and write the solution in interval notation.

Solution:

Simplify each side as much as possible.		$8x-2\left(5-x\right)<4\left(x+9\right)+6x$
Distribute.		$8x-10+2x<4x+36+6x$
Combine like terms.		$10x-10<10x+36$
Subtract 10*x* from both sides to collect the variables on the left.		$10x-10-10x<10x+36-10x$
Simplify.		$-10<36$

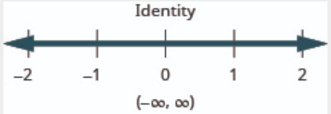
The *x*’s are gone, and we have a true statement.		The inequality is an identity. The solution is all real numbers.
Graph the solution on the number line.		
Write the solution in interval notation.		$(-\infty, \infty)$

Note:
Exercise:

Problem:

Solve the inequality $4b - 3\left(3 - b\right) > 5\left(b - 6\right) + 2b$, graph the solution on the number line, and write the solution in interval notation.

Solution:

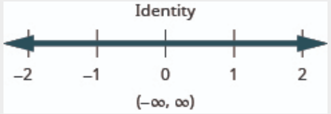


Note:
Exercise:

Problem:

Solve the inequality $9h - 7\left(2 - h\right) < 8\left(h + 11\right) + 8h$, graph the solution on the number line, and write the solution in interval notation.

Solution:



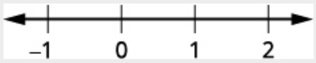
We can clear fractions in inequalities much as we did in equations. Again, be careful with the signs when multiplying or dividing by a negative.

Example:
Exercise:

Problem:

Solve the inequality $\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$, graph the solution on the number line, and write the solution in interval notation.

Solution:

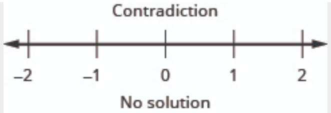
	$\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$
Multiply both sides by the LCD, 24, to clear the fractions.	$24\left(\frac{1}{3}a - \frac{1}{8}a\right) > 24\left(\frac{5}{24}a + \frac{3}{4}\right)$
Simplify.	$8a - 3a > 5a + 18$
Combine like terms.	$5a > 5a + 18$
Subtract $5a$ from both sides to collect the variables on the left.	$5a - 5a > 5a - 5a + 18$
Simplify.	$0 > 18$
The statement is false.	The inequality is a contradiction. There is no solution.
Graph the solution on the number line.	
Write the solution in interval notation.	There is no solution.

Note:
Exercise:

Problem:

Solve the inequality $\frac{1}{4}x - \frac{1}{12}x > \frac{1}{6}x + \frac{7}{8}$, graph the solution on the number line, and write the solution in interval notation.

Solution:

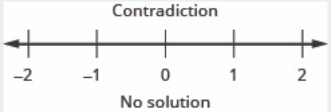


Note:
Exercise:

Problem:

Solve the inequality $\frac{2}{5}z - \frac{1}{3}z < \frac{1}{15}z - \frac{3}{5}$, graph the solution on the number line, and write the solution in interval notation.

Solution:



Translate to an Inequality and Solve

To translate English sentences into inequalities, we need to recognize the phrases that indicate the inequality. Some words are easy, like “more than” and “less than.” But others are not as obvious. [link] shows some common phrases that indicate inequalities.

$>$	$\geq$	$<$	$\leq$
is greater than	is greater than or equal to	is less than	is less than or equal to
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

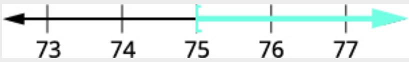
Example:
Exercise:

Problem:

Translate and solve. Then graph the solution on the number line, and write the solution in interval notation.
Equation:

$$\text{Twenty-seven less than } x \text{ is at least } 48.$$

Solution:

	Twenty-seven less than x $\underbrace{\text{is at least}}$ 48.
Translate.	$x - 27 \geq 48$
Solve—add 27 to both sides.	$x - 27 + 27 \geq 48 + 27$
Simplify.	$x \geq 75$
Graph on the number line.	
Write in interval notation.	$[75, \infty)$

Note:
Exercise:

Problem:

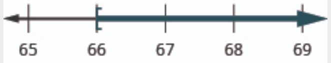
Translate and solve. Then graph the solution on the number line, and write the solution in interval notation.

Nineteen less than p is no less than 47.

Solution:

$p - 19 \geq 47$

$p \geq 66$



$[66, \infty)$

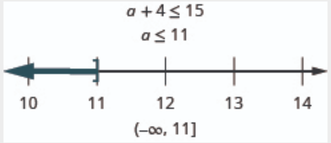
Note:
Exercise:

Problem:

Translate and solve. Then graph the solution on the number line, and write the solution in interval notation.

Four more than *a* is at most 15.

Solution:



Solve Applications with Linear Inequalities

Many real-life situations require us to solve inequalities. The method we will use to solve applications with linear inequalities is very much like the one we used when we solved applications with equations.

We will read the problem and make sure all the words are understood. Next, we will identify what we are looking for and assign a variable to represent it. We will restate the problem in one sentence to make it easy to translate into an inequality. Then, we will solve the inequality.

Sometimes an application requires the solution to be a whole number, but the algebraic solution to the inequality is not a whole number. In that case, we must round the algebraic solution to a whole number. The context of the application will determine whether we round up or down.

Example:
Exercise:

Problem:

Dawn won a mini-grant of $4,000 to buy tablet computers for her classroom. The tablets she would like to buy cost $254.12 each, including tax and delivery. What is the maximum number of tablets Dawn can buy?

Solution:

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the maximum number of tablets Dawn
Step 3. Name what you are looking for.	
Choose a variable to represent that quantity.	Let $n =$ the number of tablets.
Step 4. Translate. Write a sentence that gives the information to find it.	$254.12 times the number of tablets is no more than $4,000.
Translate into an inequality.	$254.12n \leq 4000$
Step 5. Solve the inequality.	
But n must be a whole number of tablets, so round to 15.	$n \leq 15.74$ $n \leq 15$
Step 6. Check the answer in the problem and make sure it makes sense.	
Rounding down the price to $250, 15 tablets would cost $3,750, while 16 tablets would be $4,000. So a maximum of 15 tablets at $254.12 seems reasonable.	
Step 7. Answer the question with a complete sentence.	Dawn can buy a maximum of 15 tablets

Note:
Exercise:

Problem:

Angie has $20 to spend on juice boxes for her son’s preschool picnic. Each pack of juice boxes costs $2.63. What is the maximum number of packs she can buy?

Solution:

Angie can buy 7 packs of juice.

Note:
Exercise:

Problem:

Daniel wants to surprise his girlfriend with a birthday party at her favorite restaurant. It will cost $42.75 per person for dinner, including tip and tax. His budget for the party is $500. What is the maximum number of people Daniel can have at the party?

Solution:

Daniel can have 11 people at the party.

Example:
Exercise:

Problem:

Taleisha's phone plan costs her $28.80 a month plus $0.20 per text message. How many text messages can she send/receive and keep her monthly phone bill no more than $50?

Solution:

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the number of text messages Taleisha can
Step 3. Name what you are looking for. Choose a variable to represent that quantity.	Let $t =$ the number of text messages.
Step 4. Translate Write a sentence that gives the information to find it.	$28.80 plus $0.20 times the number of text messages is less than or equal to $50.
Translate into an inequality.	$28.80 + 0.20t \leq 50$
Step 5. Solve the inequality.	$0.2t \leq 21.2$ $t \leq 106$ text messages
Step 6. Check the answer in the problem and make sure it makes sense. Yes, $28.80 + 0.20\,(106) = 50$.	
Step 7. Write a sentence that answers the question.	Taleisha can send/receive no more than 106 text messages to keep her bill no more than $50.

Note:
Exercise:

Problem:

Sergio and Lizeth have a very tight vacation budget. They plan to rent a car from a company that charges $75 a week plus $0.25 a mile. How many miles can they travel during the week and still keep within their $200 budget?

Solution:

Sergio and Lizeth can travel no more than 500 miles.

Note:
Exercise:

Problem:

Rameen's heating bill is $5.42 per month plus $1.08 per therm. How many therms can Rameen use if he wants his heating bill to be a maximum of $87.50.

Solution:

Rameen can use no more than 76 therms.

Profit is the money that remains when the costs have been subtracted from the revenue. In the next example, we will find the number of jobs a small businesswoman needs to do every month in order to make a certain amount of profit.

Example:
Exercise:

Problem:

Felicity has a calligraphy business. She charges \$2.50 per wedding invitation. Her monthly expenses are \$650. How many invitations must she write to earn a profit of at least \$2,800 per month?

Solution:

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the number of invitations Felicity needs to
Step 3. Name what you are looking for.	Let $j =$ the number of invitations.
Choose a variable to represent it.	
Step 4. Translate. Write a sentence that gives the information to find it.	\$2.50 times the number of invitations minus \$650 is at least \$2,800.
Translate into an inequality.	$2.50j - 650 \geq 2{,}800$
Step 5. Solve the inequality.	$2.5j \geq 3{,}450$ $j \geq 1{,}380$ invitations
Step 6. Check the answer in the problem and make sure it makes sense.	
If Felicity wrote 1400 invitations, her profit would be $2.50\,(1400) - 650,$ or \$2,850. This is more than \$2800.	
Step 7. Write a sentence that answers the question.	Felicity must write at least 1,380 invitatio

Note:
Exercise:

Problem:

Caleb has a pet sitting business. He charges \$32 per hour. His monthly expenses are \$2,272. How many hours must he work in order to earn a profit of at least \$800 per month?

Solution:

Caleb must work at least 96 hours.

Note:
Exercise:

Problem:

Elliot has a landscape maintenance business. His monthly expenses are $1,100. If he charges $60 per job, how many jobs must he do to earn a profit of at least $4,000 a month?

Solution:

Elliot must work at least 85 jobs.

There are many situations in which several quantities contribute to the total expense. We must make sure to account for all the individual expenses when we solve problems like this.

Example:
Exercise:

Problem:

Malik is planning a six-day summer vacation trip. He has $840 in savings, and he earns $45 per hour for tutoring. The trip will cost him $525 for airfare, $780 for food and sightseeing, and $95 per night for the hotel. How many hours must he tutor to have enough money to pay for the trip?

Solution:

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the number of hours Malik must tutor
Step 3. Name what you are looking for. Choose a variable to represent that quantity.	Let $h =$ the number of hours.
Step 4. Translate. Write a sentence that gives the information to find it.	The expenses must be less than or equal to the income. The cost of airfare plus the cost of food and sightseeing and the hotel bill must be less than the savings plus the amount earned tutoring.
Translate into an inequality.	$525 + 780 + 95\,(6) \leq 840 + 45h$
Step 5. Solve the inequality.	$1{,}875 \leq 840 + 45h$ $1{,}035 \leq 45h$ $23 \leq h$ $h \geq 23$
Step 6. Check the answer in the problem and make sure it makes sense. We substitute 23 into the inequality. $1{,}875 \leq 840 + 45h$ $1{,}875 \leq 840 + 45\,(23)$ $1{,}875 \leq 1875$	
Step 7. Write a sentence that answers the question.	Malik must tutor at least 23 hours.

Note:
Exercise:

Problem:

Brenda’s best friend is having a destination wedding and the event will last three days. Brenda has $500 in savings and can earn $15 an hour babysitting. She expects to pay $350 airfare, $375 for food and entertainment and $60 a night for her share of a hotel room. How many hours must she babysit to have enough money to pay for the trip?

Solution:

Brenda must babysit at least 27 hours.

Note:
Exercise:

Problem:

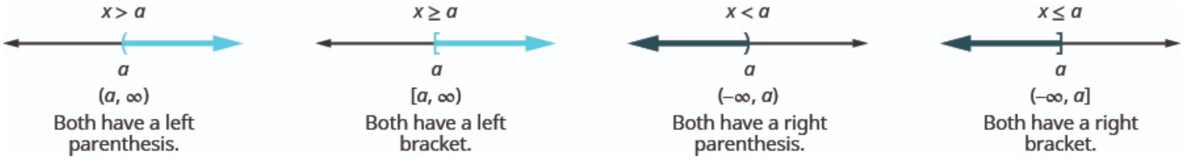
Josue wants to go on a 10-night road trip with friends next spring. It will cost him $180 for gas, $450 for food, and $49 per night to share a motel room. He has $520 in savings and can earn $30 per driveway shoveling snow. How many driveways must he shovel to have enough money to pay for the trip?

Solution:

Josue must shovel at least 20 driveways.

Key Concepts

- **Inequalities, Number Lines, and Interval Notation**
  $x > a$ $x \geq a$ $x < a$ $x \leq a$



x > a
a
(a, ∞)
Both have a left parenthesis.
x ≥ a
a
[a, ∞)
Both have a left bracket.
x < a
a
(−∞, a)
Both have a right parenthesis.
x ≤ a
a
(−∞, a]
Both have a right bracket.

- **Linear Inequality**
  - A **linear inequality** is an inequality in one variable that can be written in one of the following forms where a, b, and c are real numbers and $a \neq 0$:
    Equation:

$$ax + b < c, \qquad ax + b \leq c, \qquad ax + b > c, \qquad ax + b \geq c.$$

- **Addition and Subtraction Property of Inequality**
  - For any numbers a, b, and c, if $a < b$, then
    Equation:

$$\begin{array}{ll} a + c < b + c & a - c < b - c \\ a + c > b + c & a - c > b - c \end{array}$$

  - We can add or subtract the same quantity from both sides of an inequality and still keep the inequality.
- **Multiplication and Division Property of Inequality**
  - For any numbers a, b, and c,
    multiply or divide by a **positive**

$$\text{if } a < b \text{ and } c > 0, \text{ then } ac < bc \text{ and } \frac{a}{c} < \frac{b}{c}.$$
$$\text{if } a > b \text{ and } c > 0, \text{ then } ac > bc \text{ and } \frac{a}{c} > \frac{b}{c}.$$

multiply or divide by a **negative**

$$\text{if } a < b \text{ and } c < 0, \text{ then } ac > bc \text{ and } \frac{a}{c} > \frac{b}{c}.$$
$$\text{if } a > b \text{ and } c < 0, \text{ then } ac < bc \text{ and } \frac{a}{c} < \frac{b}{c}.$$

- **Phrases that indicate inequalities**

⩾	≧	⩽	≦
is greater than	is greater than or equal to	is less than	is less than or equal to
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

Practice Makes Perfect

Graph Inequalities on the Number Line

In the following exercises, graph each inequality on the number line and write in interval notation.

Exercise:

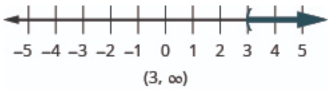
Problem:
ⓐ $x < -2$
ⓑ $x \geq -3.5$
ⓒ $x \leq \frac{2}{3}$

Exercise:

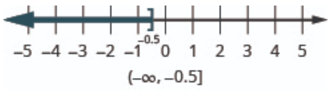
Problem:
ⓐ $x > 3$
ⓑ $x \leq -0.5$
ⓒ $x \geq \frac{1}{3}$

Solution:

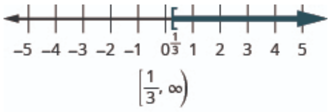
ⓐ



ⓑ



ⓒ



Exercise:

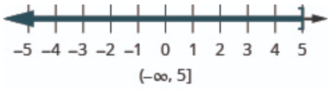
ⓐ $x \geq -4$
ⓑ $x < 2.5$
Problem: ⓒ $x > -\frac{3}{2}$

Exercise:

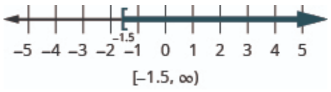
ⓐ $x \leq 5$
ⓑ $x \geq -1.5$
Problem: ⓒ $x < -\frac{7}{3}$

Solution:

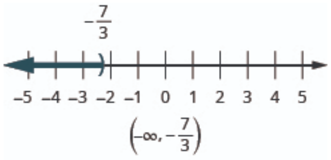
ⓐ


–5 –4 –3 –2 –1 0 1 2 3 4 5
(–∞, 5]

ⓑ


–1.5
–5 –4 –3 –2 –1 0 1 2 3 4 5
[–1.5, ∞)

ⓒ


–7/3
–5 –4 –3 –2 –1 0 1 2 3 4 5
(–∞, –7/3)

Exercise:

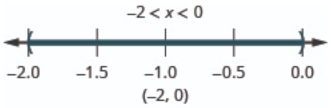
ⓐ $-5 < x < 2$
ⓑ $-3 \leq x < -1$
Problem: ⓒ $0 \leq x \leq 1.5$

Exercise:

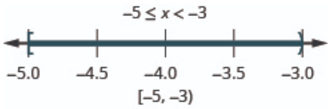
ⓐ $-2 < x < 0$
ⓑ $-5 \leq x < -3$
Problem: ⓒ $0 \leq x \leq 3.5$

Solution:

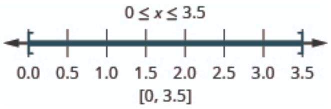
ⓐ


–2 < x < 0
–2.0
–1.5
–1.0
–0.5
0.0
(–2, 0)

ⓑ


–5 ≤ x < –3
–5.0
–4.5
–4.0
–3.5
–3.0
[–5, –3)

ⓒ


0 ≤ x ≤ 3.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
[0, 3.5]

Exercise:

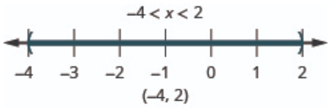
ⓐ $-1 < x < 3$
ⓑ $-3 < x \leq -2$
Problem: ⓒ $-1.25 \leq x \leq 0$

Exercise:

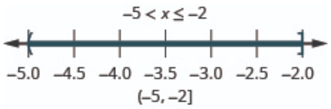
ⓐ $-4 < x < 2$
ⓑ $-5 < x \leq -2$
Problem: ⓒ $-3.75 \leq x \leq 0$

Solution:

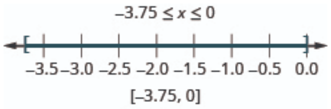
ⓐ


–4 < x < 2
–4
–3
–2
–1
0
1
2
(–4, 2)

ⓑ


–5 < x ≤ –2
–5.0
–4.5
–4.0
–3.5
–3.0
–2.5
–2.0
(–5, –2]

ⓒ


–3.75 ≤ x ≤ 0
–3.5
–3.0
–2.5
–2.0
–1.5
–1.0
–0.5
0.0
[–3.75, 0]

Solve Linear Inequalities

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

Exercise:

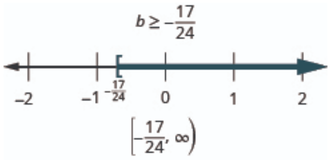
ⓐ $a+\frac{3}{4}\ge\frac{7}{10}$
ⓑ $8x>72$
Problem: ⓒ $20>\frac{2}{5}h$

Exercise:

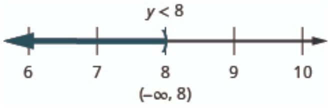
ⓐ $b+\frac{7}{8}\ge\frac{1}{6}$
ⓑ $6y<48$
Problem: ⓒ $40<\frac{5}{8}k$

Solution:

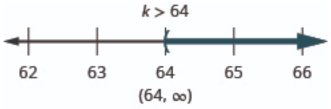
ⓐ


$b\ge-\frac{17}{24}$
–2
–1
$-\frac{17}{24}$
0
1
2
$\left[-\frac{17}{24},\infty\right)$

ⓑ



ⓒ


$k>64$
62
63
64
65
66
(64, ∞)

Exercise:

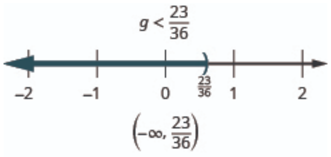
ⓐ $f-\frac{13}{20}<-\frac{5}{12}$
ⓑ $9t\ge-27$
Problem: ⓒ $\frac{7}{6}j\ge42$

Exercise:

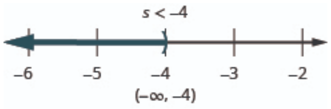
ⓐ $g - \frac{11}{12} < -\frac{5}{18}$
ⓑ $7s < -28$
Problem: ⓒ $\frac{9}{4}g \leq 36$

Solution:

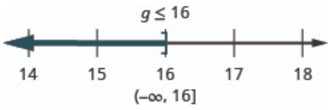
ⓐ


$g < \frac{23}{36}$
–2
–1
0
$\frac{23}{36}$
1
2
$\left(-\infty, \frac{23}{36}\right)$

ⓑ


$s < -4$
–6
–5
–4
–3
–2
$(-\infty, -4)$

ⓒ


$g \leq 16$
14
15
16
17
18
$(-\infty, 16]$

Exercise:

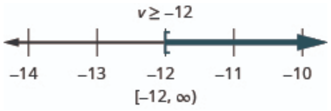
ⓐ $-5u \geq 65$
Problem: ⓑ $\frac{a}{-3} \leq 9$

Exercise:

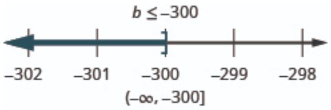
ⓐ $-8v \leq 96$
Problem: ⓑ $\frac{b}{-10} \geq 30$

Solution:

ⓐ


$v \geq -12$
–14
–13
–12
–11
–10
$[-12, \infty)$

ⓑ



b ≤ –300
–302
–301
–300
–299
–298
(–∞, –300]

Exercise:

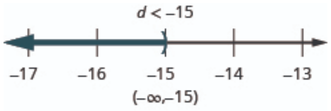
Problem: ⓐ $-9c < 126$ ⓑ $-25 < \frac{p}{-5}$

Exercise:

Problem: ⓐ $-7d > 105$ ⓑ $-18 > \frac{q}{-6}$

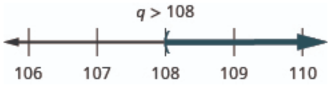
Solution:

ⓐ



d < –15
–17
–16
–15
–14
–13
(–∞,–15)

ⓑ



q > 108
106
107
108
109
110

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

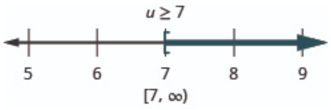
Exercise:

Problem: $4v \geq 9v - 40$

Exercise:

Problem: $5u \leq 8u - 21$

Solution:



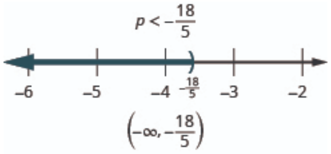
Exercise:

Problem: $13q < 7q - 29$

Exercise:

Problem: $9p > 14p - 18$

Solution:


$p < -\frac{18}{5}$
–6
–5
–4
$-\frac{18}{5}$
–3
–2
$\left(-\infty, -\frac{18}{5}\right)$

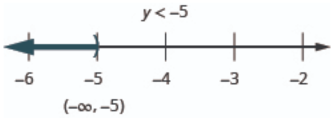
Exercise:

Problem: $12x + 3\left(x + 7\right) > 10x - 24$

Exercise:

Problem: $9y + 5\left(y + 3\right) < 4y - 35$

Solution:



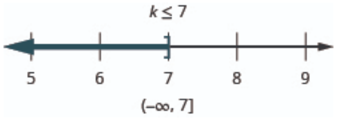
Exercise:

Problem: $6h - 4\left(h - 1\right) \leq 7h - 11$

Exercise:

Problem: $4k - \left(k - 2\right) \geq 7k - 26$

Solution:



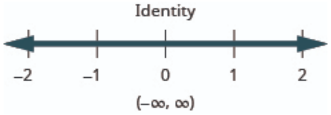
Exercise:

Problem: $8m - 2\left(14 - m\right) \geq 7\left(m - 4\right) + 3m$

Exercise:

Problem: $6n - 12\left(3 - n\right) \leq 9\left(n - 4\right) + 9n$

Solution:


Identity
–2
–1
0
1
2
(–∞, ∞)

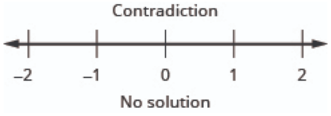
Exercise:

Problem: $\frac{3}{4}b-\frac{1}{3}b<\frac{5}{12}b-\frac{1}{2}$

Exercise:

Problem: $9u+5\left(2u-5\right)\ge 12\left(u-1\right)+7u$

Solution:


Contradiction
–2
–1
0
1
2
No solution

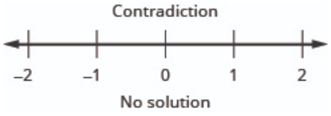
Exercise:

Problem: $\frac{2}{3}g-\frac{1}{2}\left(g-14\right)\le \frac{1}{6}\left(g+42\right)$

Exercise:

Problem: $\frac{4}{5}h-\frac{2}{3}\left(h-9\right)\ge \frac{1}{15}\left(2h+90\right)$

Solution:


Contradiction
–2
–1
0
1
2
No solution

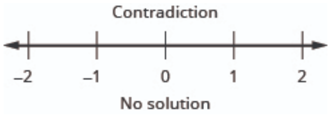
Exercise:

Problem: $\frac{5}{6}a-\frac{1}{4}a>\frac{7}{12}a+\frac{2}{3}$

Exercise:

Problem: $12v+3\left(4v-1\right)\le 19\left(v-2\right)+5v$

Solution:


Contradiction
–2
–1
0
1
2
No solution

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

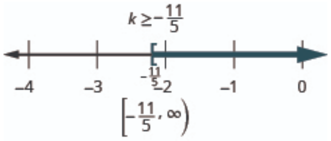
Exercise:

Problem: $15k \leq -40$

Exercise:

Problem: $35k \geq -77$

Solution:


$k \geq -\frac{11}{5}$
–4
–3
$-\frac{11}{5}$
–2
–1
0
$\left[-\frac{11}{5}, \infty\right)$

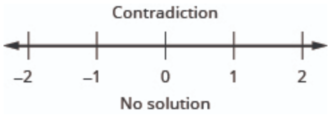
Exercise:

Problem: $23p - 2\left(6 - 5p\right) > 3\left(11p - 4\right)$

Exercise:

Problem: $18q - 4\left(10 - 3q\right) < 5\left(6q - 8\right)$

Solution:


Contradiction
–2
–1
0
1
2
No solution

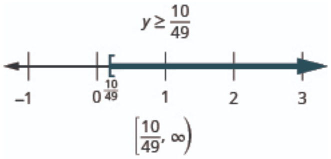
Exercise:

Problem: $-\frac{9}{4}x \geq -\frac{5}{12}$

Exercise:

Problem: $-\frac{21}{8}y \leq -\frac{15}{28}$

Solution:


$y \geq \frac{10}{49}$
–1
0
$\frac{10}{49}$
1
2
3
$\left[\frac{10}{49}, \infty\right)$

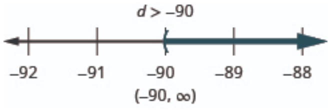
Exercise:

Problem: $c + 34 < -99$

Exercise:

Problem: $d + 29 > -61$

Solution:


d > –90
–92
–91
–90
–89
–88
(–90, ∞)

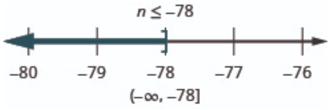
Exercise:

Problem: $\frac{m}{18} \geq -4$

Exercise:

Problem: $\frac{n}{13} \leq -6$

Solution:


n ≤ –78
–80
–79
–78
–77
–76
(–∞, –78]

Translate to an Inequality and Solve

In the following exercises, translate and solve. Then graph the solution on the number line and write the solution in interval notation.

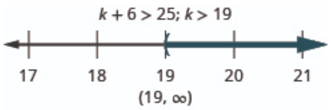
Exercise:

Problem: Three more than *h* is no less than 25.

Exercise:

Problem: Six more than *k* exceeds 25.

Solution:



k + 6 > 25; k > 19
17
18
19
20
21
(19, ∞)

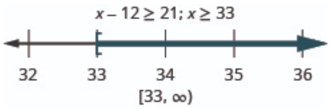
Exercise:

Problem: Ten less than *w* is at least 39.

Exercise:

Problem: Twelve less than *x* is no less than 21.

Solution:



x – 12 ≥ 21; x ≥ 33
32
33
34
35
36
[33, ∞)

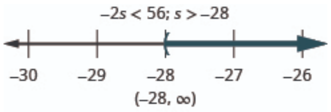
Exercise:

Problem: Negative five times *r* is no more than 95.

Exercise:

Problem: Negative two times *s* is lower than 56.

Solution:



–2s < 56; s > –28
–30
–29
–28
–27
–26
(–28, ∞)

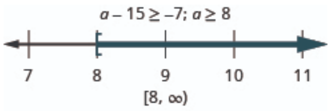
Exercise:

Problem: Nineteen less than *b* is at most -22.

Exercise:

Problem: Fifteen less than *a* is at least -7.

Solution:



a – 15 ≥ –7; a ≥ 8
7
8
9
10
11
[8, ∞)

Solve Applications with Linear Inequalities

In the following exercises, solve.

Exercise:

Problem:

Alan is loading a pallet with boxes that each weighs 45 pounds. The pallet can safely support no more than 900 pounds. How many boxes can he safely load onto the pallet?

Exercise:

Problem:

The elevator in Yehire’s apartment building has a sign that says the maximum weight is 2100 pounds. If the average weight of one person is 150 pounds, how many people can safely ride the elevator?

Solution:

A maximum of 14 people can safely ride in the elevator.

Exercise:

Problem:

Andre is looking at apartments with three of his friends. They want the monthly rent to be no more than $2,360. If the roommates split the rent evenly among the four of them, what is the maximum rent each will pay?

Exercise:

Problem:

Arleen got a $20 gift card for the coffee shop. Her favorite iced drink costs $3.79. What is the maximum number of drinks she can buy with the gift card?

Solution:

five drinks

Exercise:

Problem:

Teegan likes to play golf. He has budgeted $60 next month for the driving range. It costs him $10.55 for a bucket of balls each time he goes. What is the maximum number of times he can go to the driving range next month?

Exercise:

Problem:

Ryan charges his neighbors $17.50 to wash their car. How many cars must he wash next summer if his goal is to earn at least $1,500?

Solution:

86 cars

Exercise:

Problem:

Keshad gets paid $2,400 per month plus 6% of his sales. His brother earns $3,300 per month. For what amount of total sales will Keshad’s monthly pay be higher than his brother’s monthly pay?

Exercise:

Problem:

Kimuyen needs to earn $4,150 per month in order to pay all her expenses. Her job pays her $3,475 per month plus 4% of her total sales. What is the minimum Kimuyen's total sales must be in order for her to pay all her expenses?

Solution:

$16,875

Exercise:

Problem:

Andre has been offered an entry-level job. The company offered him $48,000 per year plus 3.5% of his total sales. Andre knows that the average pay for this job is $62,000. What would Andre's total sales need to be for his pay to be at least as high as the average pay for this job?

Exercise:

Problem:

Nataly is considering two job offers. The first job would pay her $83,000 per year. The second would pay her $66,500 plus 15% of her total sales. What would her total sales need to be for her salary on the second offer be higher than the first?

Solution:

$110,000

Exercise:

Problem:

Jake's water bill is $24.80 per month plus $2.20 per ccf (hundred cubic feet) of water. What is the maximum number of ccf Jake can use if he wants his bill to be no more than $60?

Exercise:

Problem:

Kiyoshi's phone plan costs $17.50 per month plus $0.15 per text message. What is the maximum number of text messages Kiyoshi can use so the phone bill is no more than $56.60?

Solution:

260 messages

Exercise:

Problem:

Marlon's TV plan costs $49.99 per month plus $5.49 per first-run movie. How many first-run movies can he watch if he wants to keep his monthly bill to be a maximum of $100?

Exercise:

Problem:

Kellen wants to rent a banquet room in a restaurant for her cousin's baby shower. The restaurant charges $350 for the banquet room plus $32.50 per person for lunch. How many people can Kellen have at the shower if she wants the maximum cost to be $1,500?

Solution:

35 people

Exercise:

Problem:

Moshde runs a hairstyling business from her house. She charges $45 for a haircut and style. Her monthly expenses are $960. She wants to be able to put at least $1,200 per month into her savings account order to open her own salon. How many “cut & styles” must she do to save at least $1,200 per month?

Exercise:

Problem:

Noe installs and configures software on home computers. He charges $125 per job. His monthly expenses are $1,600. How many jobs must he work in order to make a profit of at least $2,400?

Solution:

32 jobs

Exercise:

Problem:

Katherine is a personal chef. She charges $115 per four-person meal. Her monthly expenses are $3,150. How many four-person meals must she sell in order to make a profit of at least $1,900?

Exercise:

Problem:

Melissa makes necklaces and sells them online. She charges $88 per necklace. Her monthly expenses are $3,745. How many necklaces must she sell if she wants to make a profit of at least $1,650?

Solution:

62 necklaces

Exercise:

Problem:

Five student government officers want to go to the state convention. It will cost them $110 for registration, $375 for transportation and food, and $42 per person for the hotel. There is $450 budgeted for the convention in the student government savings account. They can earn the rest of the money they need by having a car wash. If they charge $5 per car, how many cars must they wash in order to have enough money to pay for the trip?

Exercise:

Problem:

Cesar is planning a four-day trip to visit his friend at a college in another state. It will cost him $198 for airfare, $56 for local transportation, and $45 per day for food. He has $189 in savings and can earn $35 for each lawn he mows. How many lawns must he mow to have enough money to pay for the trip?

Solution:

seven lawns

Exercise:

Problem:

Alonzo works as a car detailer. He charges $175 per car. He is planning to move out of his parents’ house and rent his first apartment. He will need to pay $120 for application fees, $950 for security deposit, and first and last months’ rent at $1,140 per month. He has $1,810 in savings. How many cars must he detail to have enough money to rent the apartment?

Exercise:

Problem:

Eun-Kyung works as a tutor and earns $60 per hour. She has $792 in savings. She is planning an anniversary party for her parents. She would like to invite 40 guests. The party will cost her $1,520 for food and drinks and $150 for the photographer. She will also have a favor for each of the guests, and each favor will cost $7.50. How many hours must she tutor to have enough money for the party?

Solution:

20 hours

Everyday Math

Exercise:

Problem:

Maximum load on a stage In 2014, a high school stage collapsed in Fullerton, California, when 250 students got on stage for the finale of a musical production. Two dozen students were injured. The stage could support a maximum of 12,750 pounds. If the average weight of a student is assumed to be 140 pounds, what is the maximum number of students who could safely be on the stage?

Exercise:

Problem:

Maximum weight on a boat In 2004, a water taxi sank in Baltimore harbor and five people drowned. The water taxi had a maximum capacity of 3,500 pounds (25 people with average weight 140 pounds). The average weight of the 25 people on the water taxi when it sank was 168 pounds per person. What should the maximum number of people of this weight have been?

Solution:

20 people

Exercise:

Problem:

Wedding budget Adele and Walter found the perfect venue for their wedding reception. The cost is $9850 for up to 100 guests, plus $38 for each additional guest. How many guests can attend if Adele and Walter want the total cost to be no more than $12,500?

Exercise:

Problem:

Shower budget Penny is planning a baby shower for her daughter-in-law. The restaurant charges $950 for up to 25 guests, plus $31.95 for each additional guest. How many guests can attend if Penny wants the total cost to be no more than $1,500?

Solution:

42 guests

Writing Exercises

Exercise:

Problem: Explain why it is necessary to reverse the inequality when solving $-5x > 10$.

Exercise:

Problem: Explain why it is necessary to reverse the inequality when solving $\frac{n}{-3} < 12$.

Solution:

Answers will vary.

Exercise:

Problem:

Find your last month’s phone bill and the hourly salary you are paid at your job. Calculate the number of hours of work it would take you to earn at least enough money to pay your phone bill by writing an appropriate inequality and then solving it. Do you feel this is an appropriate number of hours? Is this the appropriate phone plan for you?

Exercise:

Problem:

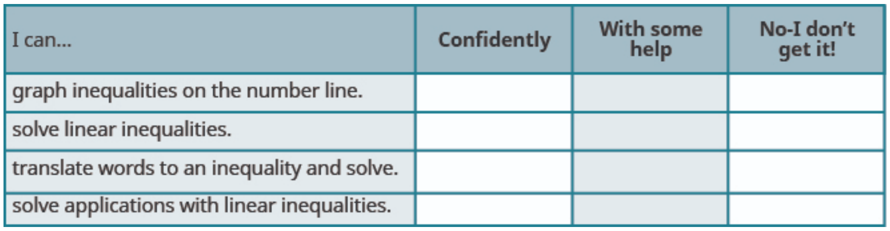
Find out how many units you have left, after this term, to achieve your college goal and estimate the number of units you can take each term in college. Calculate the number of terms it will take you to achieve your college goal by writing an appropriate inequality and then solving it. Is this an acceptable number of terms until you meet your goal? What are some ways you could accelerate this process?

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don’t get it!
graph inequalities on the number line.			
solve linear inequalities.			
translate words to an inequality and solve.			
solve applications with linear inequalities.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Solve Linear Inequalities

By the end of this section, you will be able to:

- Graph inequalities on the number line
- Solve inequalities using the Subtraction and Addition Properties of inequality
- Solve inequalities using the Division and Multiplication Properties of inequality
- Solve inequalities that require simplification
- Translate to an inequality and solve

Note:
Before you get started, take this readiness quiz.

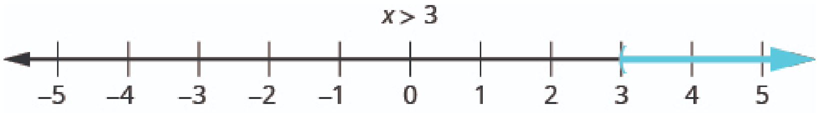
1. Translate from algebra to English: $15 > x$.
   If you missed this problem, review [link].
2. Solve: $n - 9 = -42$.
   If you missed this problem, review [link].
3. Solve: $-5p = -23$.
   If you missed this problem, review [link].
4. Solve: $3a - 12 = 7a - 20$.
   If you missed this problem, review [link].

Graph Inequalities on the Number Line

Do you remember what it means for a number to be a solution to an equation? A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

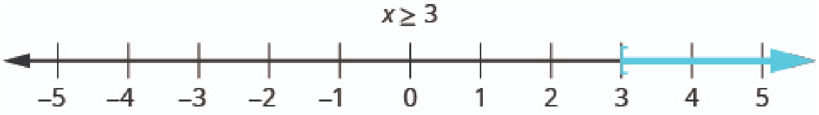
What about the solution of an inequality? What number would make the inequality $x > 3$ true? Are you thinking, '*x* could be 4'? That's correct, but *x* could be 5 too, or 20, or even 3.001. Any number greater than 3 is a solution to the inequality $x > 3$.

We show the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of 3, to show that all numbers greater than 3 are solutions. Because the number 3 itself is not a solution, we put an open parenthesis at 3. The graph of $x > 3$ is shown in [link]. Please note that the following convention is used: light blue arrows point in the positive direction and dark blue arrows point in the negative direction.



The inequality $x > 3$ is graphed on this number line.

The graph of the inequality $x \geq 3$ is very much like the graph of $x > 3$, but now we need to show that 3 is a solution, too. We do that by putting a bracket at $x = 3$, as shown in [link].



The inequality $x \geq 3$ is graphed on this number line.

Notice that the open parentheses symbol, (, shows that the endpoint of the inequality is not included. The open bracket symbol, [, shows that the endpoint is included.

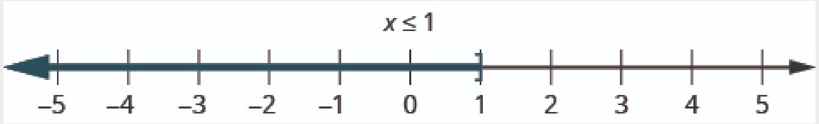
Example:
Exercise:

Problem: Graph on the number line:

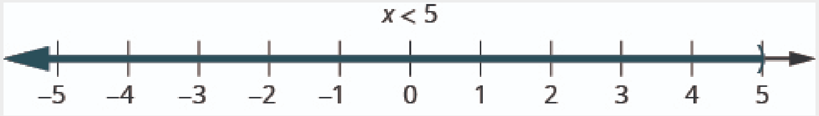
ⓐ $x \leq 1$ ⓑ $x < 5$ ⓒ $x > -1$

Solution:
Solution

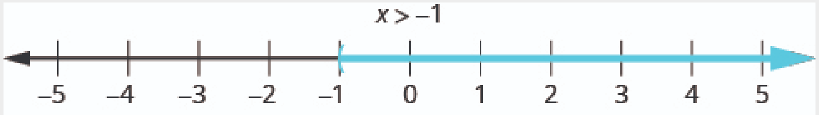
ⓐ $x \leq 1$
This means all numbers less than or equal to 1. We shade in all the numbers on the number line to the left of 1 and put a bracket at $x = 1$ to show that it is included.



ⓑ $x < 5$
This means all numbers less than 5, but not including 5. We shade in all the numbers on the number line to the left of 5 and put a parenthesis at $x = 5$ to show it is not included.


x < 5
–5
–4
–3
–2
–1
0
1
2
3
4
5

ⓒ $x > -1$
This means all numbers greater than -1, but not including -1. We shade in all the numbers on the number line to the right of -1, then put a parenthesis at $x = -1$ to show it is not included.

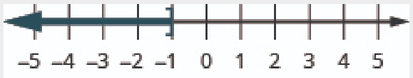

x > –1
–5
–4
–3
–2
–1
0
1
2
3
4
5

Note:
Exercise:

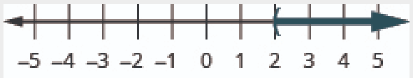
Problem: Graph on the number line: ⓐ $x \leq -1$ ⓑ $x > 2$ ⓒ $x < 3$

Solution:

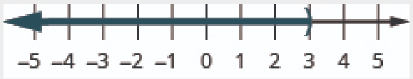
ⓐ


–5
–4
–3
–2
–1
0
1
2
3
4
5

ⓑ


–5
–4
–3
–2
–1
0
1
2
3
4
5

ⓒ

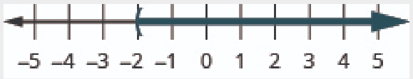

–5
–4
–3
–2
–1
0
1
2
3
4
5

Note:
Exercise:

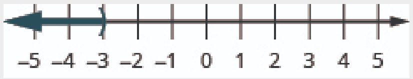
Problem: Graph on the number line: ⓐ $x > -2$ ⓑ $x < -3$ ⓒ $x \geq -1$

Solution:

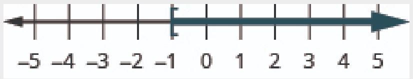
ⓐ



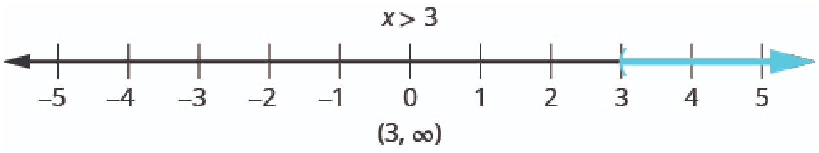
ⓑ



ⓒ

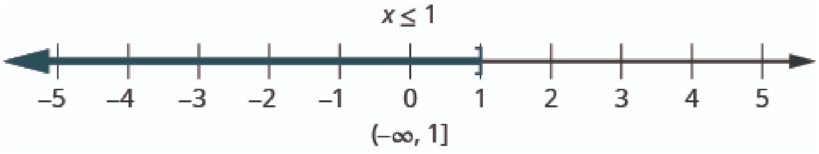


We can also represent inequalities using *interval notation.* As we saw above, the inequality $x > 3$ means all numbers greater than 3. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as ‘infinity’. It is not an actual number. [link] shows both the number line and the interval notation.



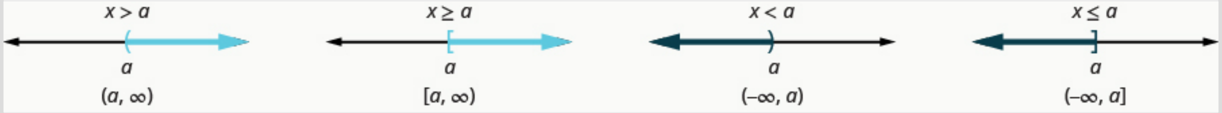
The inequality $x > 3$ is graphed on this number line and written in interval notation.

The inequality $x \leq 1$ means all numbers less than or equal to 1. There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as ‘negative infinity’. [link] shows both the number line and interval notation.

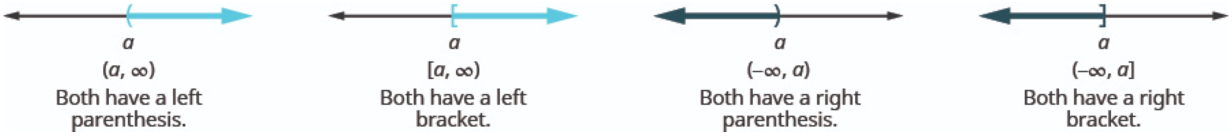

x ≤ 1
–5
–4
–3
–2
–1
0
1
2
3
4
5
(–∞, 1]

The inequality $x \le 1$ is graphed on this number line and written in interval notation.

Note:
Inequalities, Number Lines, and Interval Notation


x > a
a
(a, ∞)
x ≥ a
a
[a, ∞)
x < a
a
(–∞, a)
x ≤ a
a
(–∞, a]

Did you notice how the parenthesis or bracket in the interval notation matches the symbol at the endpoint of the arrow? These relationships are shown in [link].


a
(a, ∞)
Both have a left parenthesis.
a
[a, ∞)
Both have a left bracket.
a
(–∞, a)
Both have a right parenthesis.
a
(–∞, a]
Both have a right bracket.

The notation for inequalities on a number line and in interval notation use similar symbols to express the endpoints of intervals.

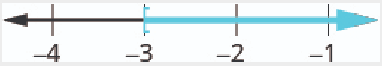
Example:
Exercise:

Problem: Graph on the number line and write in interval notation.

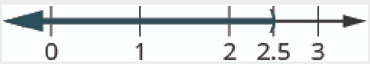
ⓐ $x \geq -3$ ⓑ $x < 2.5$ ⓒ $x \leq -\frac{3}{5}$

Solution:
Solution

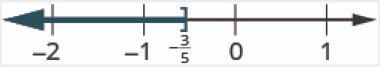
ⓐ

	$x \geq -3$
Shade to the right of -3, and put a bracket at -3.	
Write in interval notation.	$[-3, \infty)$

ⓑ

	$x < 2.5$
Shade to the left of 2.5, and put a parenthesis at 2.5.	
Write in interval notation.	$(-\infty, 2.5)$

ⓒ

	$x \le -\frac{3}{5}$
Shade to the left of $-\frac{3}{5}$, and put a bracket at $-\frac{3}{5}$.	 −2 −1 $-\frac{3}{5}$ 0 1
Write in interval notation.	$\left(-\infty, -\frac{3}{5}\right]$

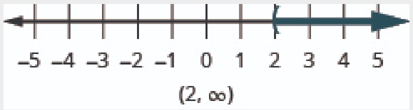
Note:
Exercise:

Problem: Graph on the number line and write in interval notation:

ⓐ $x > 2$ ⓑ $x \le -1.5$ ⓒ $x \ge \frac{3}{4}$

Solution:

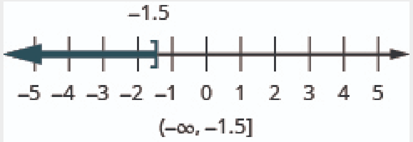
ⓐ



–5 –4 –3 –2 –1 0 1 2 3 4 5

(2, ∞)

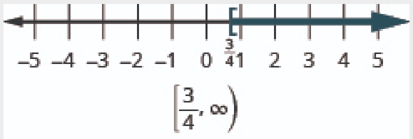
ⓑ



–1.5
–5 –4 –3 –2 –1 0 1 2 3 4 5

(–∞, –1.5]

ⓒ


–5 –4 –3 –2 –1 0 3/4 1 2 3 4 5
[3/4, ∞)

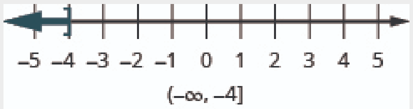
Note:
Exercise:

Problem: Graph on the number line and write in interval notation:

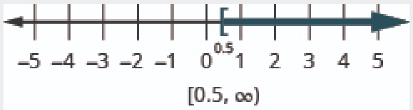
ⓐ $x \le -4$ ⓑ $x \ge 0.5$ ⓒ $x < -\frac{2}{3}$

Solution:

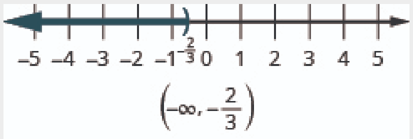
ⓐ


–5 –4 –3 –2 –1 0 1 2 3 4 5
(–∞, –4]

ⓑ


–5 –4 –3 –2 –1 0 0.5 1 2 3 4 5
[0.5, ∞)

ⓒ


–5 –4 –3 –2 –1 –2/3 0 1 2 3 4 5
(–∞, –2/3)

Solve Inequalities using the Subtraction and Addition Properties of Inequality

The Subtraction and Addition Properties of Equality state that if two quantities are equal, when we add or subtract the same amount from both quantities, the results will be equal.

Note:
Properties of Equality
Equation:

Subtraction Property of Equality

For any numbers a, b, and c,

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & a-c & = & b-c. \end{array}$$

Addition Property of Equality

For any numbers a, b, and c,

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & a+c & = & b+c. \end{array}$$

Similar properties hold true for inequalities.

For example, we know that −4 is less than 2.	$-4 < 2$
If we subtract 5 from both quantities, is the left side still less than the right side?	$-4-5 \; ? \; 2-5$
We get −9 on the left and −3 on the right.	$-9 \; ? \; -3$
And we know −9 is less than −3.	$-9 < -3$
	The inequality sign stayed the same.

Similarly we could show that the inequality also stays the same for addition.

This leads us to the Subtraction and Addition Properties of Inequality.

Note:
Properties of Inequality
Equation:

Subtraction Property of Inequality

For any numbers a, b, and c,

$$\begin{array}{lrcl} \text{if} & a & < & b \\ \text{then} & a-c & < & b-c. \end{array}$$

$$\begin{array}{lrcl} \text{if} & a & > & b \\ \text{then} & a-c & > & b-c. \end{array}$$

Addition Property of Inequality

For any numbers a, b, and c,

$$\begin{array}{lrcl} \text{if} & a & < & b \\ \text{then} & a+c & < & b+c. \end{array}$$

$$\begin{array}{lrcl} \text{if} & a & > & b \\ \text{then} & a+c & > & b+c. \end{array}$$

We use these properties to solve inequalities, taking the same steps we used to solve equations. Solving the inequality $x+5>9$, the steps would look like this:

Equation:

$$\begin{array}{lrcl} & x+5 & > & 9 \\ \text{Subtract 5 from both sides to isolate } x. & x+5-5 & > & 9-5 \\ \text{Simplify.} & x & > & 4 \end{array}$$

Any number greater than 4 is a solution to this inequality.

Example:

Exercise:

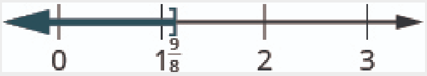
Problem:

Solve the inequality $n-\frac{1}{2} \leq \frac{5}{8}$, graph the solution on the number line, and write the solution in interval notation.

Solution:

Solution

	$n-\frac{1}{2} \leq \frac{5}{8}$
Add $\frac{1}{2}$ to both sides of the inequality.	

	$n-\frac{1}{2}+\frac{1}{2}\le\frac{5}{8}+\frac{1}{2}$
Simplify.	$n\le\frac{9}{8}$
Graph the solution on the number line.	
Write the solution in interval notation.	$\left(-\infty,\frac{9}{8}\right]$

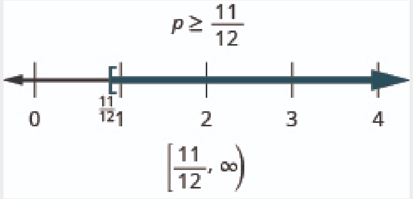
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$p-\frac{3}{4}\ge\frac{1}{6}$

Solution:



$p\ge\frac{11}{12}$

$\left[\frac{11}{12},\infty\right)$

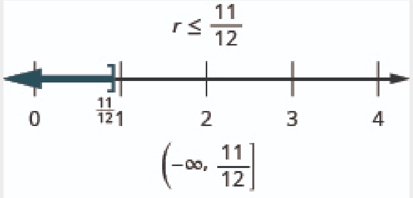
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$r - \frac{1}{3} \leq \frac{7}{12}$

Solution:



Solve Inequalities using the Division and Multiplication Properties of Inequality

The Division and Multiplication Properties of Equality state that if two quantities are equal, when we divide or multiply both quantities by the same amount, the results will also be equal (provided we don’t divide by 0).

Note:
Properties of Equality
Equation:

$$\begin{array}{ll} \textbf{Division Property of Equality} & \textbf{Multiplication Property of Equality} \\ \text{For any numbers } a, b, c, \text{and } c \neq 0, & \text{For any real numbers } a, b, c, \\ \begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & \frac{a}{c} & = & \frac{b}{c}. \end{array} & \begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & ac & = & bc. \end{array} \end{array}$$

Are there similar properties for inequalities? What happens to an inequality when we divide or multiply both sides by a constant?

Consider some numerical examples.

	$10 < 15$		$10 < 15$
Divide both sides by 5.	$\frac{10}{5} ? \frac{15}{5}$	Multiply both sides by 5.	$10(5) ? 15(5)$
Simplify.	$2 ? 3$		$50 ? 75$
Fill in the inequality signs.	$2 < 3$		$50 < 75$

Equation:

The inequality signs stayed the same.

Does the inequality stay the same when we divide or multiply by a negative number?

	$10 < 15$		$10 < 15$
Divide both sides by −5.	$\frac{10}{-5} ? \frac{15}{-5}$	Multiply both sides by −5.	$10(-5) ? 15(-5)$
Simplify.	$-2 ? -3$		$-50 ? -75$
Fill in the inequality signs.	$-2 > -3$		$-50 > -75$

Equation:

The inequality signs reversed their direction.

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses.

Here are the Division and Multiplication Properties of Inequality for easy reference.

Note:
Division and Multiplication Properties of Inequality
Equation:

$$\begin{array}{ll} \text{For any real numbers } a, b, c & \\ & \\ \text{if } a < b \text{ and } c > 0, \text{then} & \frac{a}{c} < \frac{b}{c} \text{ and } ac < bc. \\ \text{if } a > b \text{ and } c > 0, \text{then} & \frac{a}{c} > \frac{b}{c} \text{ and } ac > bc. \\ \text{if } a < b \text{ and } c < 0, \text{then} & \frac{a}{c} > \frac{b}{c} \text{ and } ac > bc. \\ \text{if } a > b \text{ and } c < 0, \text{then} & \frac{a}{c} < \frac{b}{c} \text{ and } ac < bc. \end{array}$$

When we **divide or multiply** an inequality by a:

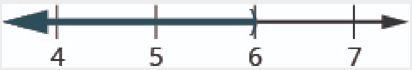
- **positive** number, the inequality stays the **same**.
- **negative** number, the inequality **reverses**.

Example:
Exercise:

Problem:

Solve the inequality $7y < 42$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

	$7y < 42$
Divide both sides of the inequality by 7. Since $7 > 0$, the inequality stays the same.	$\frac{7y}{7} < \frac{42}{7}$
Simplify.	$y < 6$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, 6)$

Note:
Exercise:

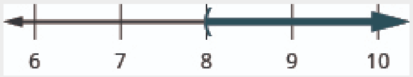
Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$9c > 72$

Solution:

$c > 8$



Note:
Exercise:

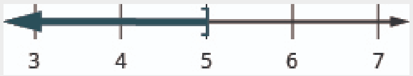
Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$12d \leq 60$

Solution:

$d \leq 5$



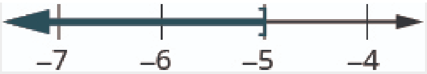
Example:
Exercise:

Problem:

Solve the inequality $-10a \geq 50$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

	$-10a \geq 50$
Divide both sides of the inequality by −10. Since $-10 < 0$, the inequality reverses.	$\frac{-10a}{-10} \leq \frac{50}{-10}$
Simplify.	$a \leq -5$
Graph the solution on the number line.	

	
Write the solution in interval notation.	(–∞, –5]

Note:
Exercise:

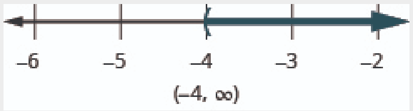
Problem:

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$-8q < 32$

Solution:

$q > -4$



(–4, ∞)

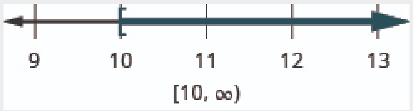
Note:
Exercise:

Problem:

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$-7r \leq -70$

Solution:



[10, ∞)

Note:
Solving Inequalities
Sometimes when solving an inequality, the variable ends up on the right. We can rewrite the inequality in reverse to get the variable to the left.
Equation:

$$x > a \text{ has the same meaning as } a < x$$

Think about it as “If Xavier is taller than Alex, then Alex is shorter than Xavier.”

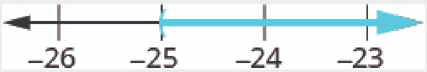
Example:
Exercise:

Problem:

Solve the inequality $-20 < \frac{4}{5}u$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

	$-20 < \frac{4}{5}u$
Multiply both sides of the inequality by $\frac{5}{4}$. Since $\frac{5}{4} > 0$, the inequality stays the same.	$\frac{5}{4}(-20) < \frac{5}{4}\left(\frac{4}{5}u\right)$
Simplify.	$-25 < u$
Rewrite the variable on the left.	$u > -25$

Graph the solution on the number line.	
Write the solution in interval notation.	(–25, ∞)

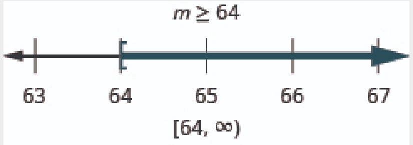
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$24 \le \frac{3}{8}m$

Solution:



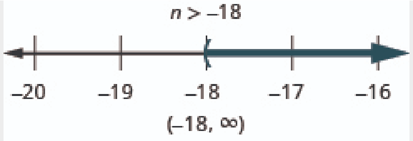
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$-24 < \frac{4}{3}n$

Solution:

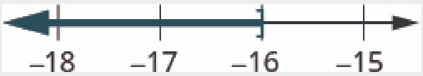


Example:
Exercise:

Problem:

Solve the inequality $\frac{t}{-2} \geq 8$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

	$\frac{t}{-2} \geq 8$
Multiply both sides of the inequality by -2. Since $-2 < 0$, the inequality reverses.	$-2\left(\frac{t}{-2}\right) \leq -2(8)$
Simplify.	$t \leq -16$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -16]$

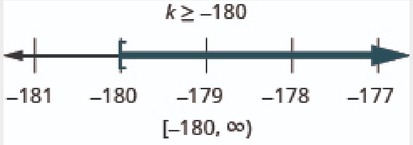
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$\frac{k}{-12} \leq 15$

Solution:



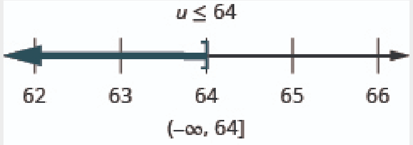
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$\frac{u}{-4} \geq -16$

Solution:



Solve Inequalities That Require Simplification

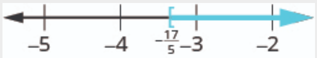
Most inequalities will take more than one step to solve. We follow the same steps we used in the general strategy for solving linear equations, but be sure to pay close attention during multiplication or division.

Example:
Exercise:

Problem:

Solve the inequality $4m \leq 9m + 17$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

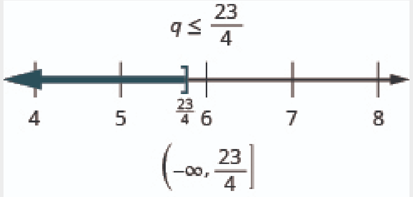
	$4m \leq 9m + 17$
Subtract $9m$ from both sides to collect the variables on the left.	$4m - 9m \leq 9m - 9m + 17$
Simplify.	$-5m \leq 17$
Divide both sides of the inequality by −5, and reverse the inequality.	$\frac{-5m}{-5} \geq \frac{17}{-5}$
Simplify.	$m \geq -\frac{17}{5}$
Graph the solution on the number line.	
Write the solution in interval notation.	$\left[-\frac{17}{5}, \infty\right)$

Note:
Exercise:

Problem:

Solve the inequality $3q \geq 7q - 23$, graph the solution on the number line, and write the solution in interval notation.

Solution:

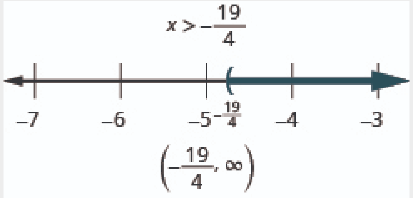

$q \leq \frac{23}{4}$
4
5
$\frac{23}{4}$
6
7
8
$\left(-\infty, \frac{23}{4}\right]$

Note:
Exercise:

Problem:

Solve the inequality $6x < 10x + 19$, graph the solution on the number line, and write the solution in interval notation.

Solution:

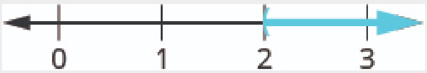

$x > -\frac{19}{4}$
–7
–6
–5
$-\frac{19}{4}$
–4
–3
$\left(-\frac{19}{4}, \infty\right)$

Example:
Exercise:

Problem:

Solve the inequality $8p + 3(p - 12) > 7p - 28$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

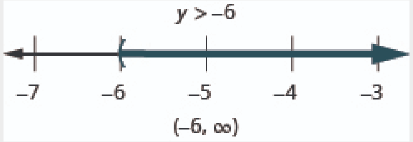
Simplify each side as much as possible.	$8p+3(p-12)>7p-28$
Distribute.	$8p+3p-36>7p-28$
Combine like terms.	$11p-36>7p-28$
Subtract $7p$ from both sides to collect the variables on the left.	$11p-36-7p>7p-28-7p$
Simplify.	$4p-36>-28$
Add 36 to both sides to collect the constants on the right.	$4p-36+36>-28+36$
Simplify.	$4p>8$
Divide both sides of the inequality by 4; the inequality stays the same.	$\frac{4p}{4}>\frac{8}{4}$
Simplify.	$p>2$
Graph the solution on the number line.	
Write the solution in interal notation.	$(2,\infty)$

Note:
Exercise:

Problem:

Solve the inequality $9y+2(y+6)>5y-24$, graph the solution on the number line, and write the solution in interval notation.

Solution:

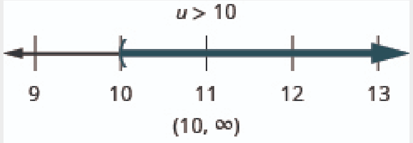


Note:
Exercise:

Problem:

Solve the inequality $6u+8(u-1)>10u+32$, graph the solution on the number line, and write the solution in interval notation.

Solution:



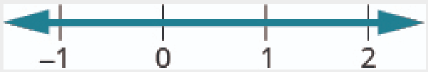
Just like some equations are identities and some are contradictions, inequalities may be identities or contradictions, too. We recognize these forms when we are left with only constants as we solve the inequality. If the result is a true statement, we have an identity. If the result is a false statement, we have a contradiction.

Example:
Exercise:

Problem:

Solve the inequality $8x-2\left(5-x\right)<4\left(x+9\right)+6x$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

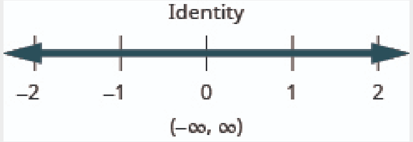
Simplify each side as much as possible.	$8x-2(5-x)<4(x+9)+6x$
Distribute.	$8x-10+2x<4x+36+6x$
Combine like terms.	$10x-10<10x+36$
Subtract $10x$ from both sides to collect the variables on the left.	$10x-10-10x<10x+36-10x$
Simplify.	$-10<36$
The x’s are gone, and we have a true statement.	The inequality is an identity. The solution is all real numbers.
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty,\infty)$

Note:
Exercise:

Problem:

Solve the inequality $4b - 3(3 - b) > 5(b - 6) + 2b$, graph the solution on the number line, and write the solution in interval notation.

Solution:

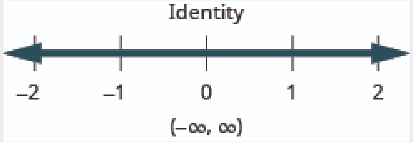


Note:
Exercise:

Problem:

Solve the inequality $9h - 7\left(2 - h\right) < 8\left(h + 11\right) + 8h$, graph the solution on the number line, and write the solution in interval notation.

Solution:

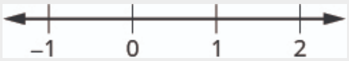


Example:
Exercise:

Problem:

Solve the inequality $\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

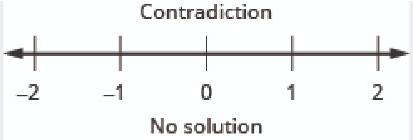
	$\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$
Multiply both sides by the LCD, 24, to clear the fractions.	$24\left(\frac{1}{3}a - \frac{1}{8}a\right) > 24\left(\frac{5}{24}a + \frac{3}{4}\right)$
Simplify.	$8a - 3a > 5a + 18$
Combine like terms.	$5a > 5a + 18$
Subtract $5a$ from both sides to collect the variables on the left.	$5a - 5a > 5a - 5a + 18$
Simplify.	$0 > 18$
The statement is false!	The inequality is a contradiction.
	There is no solution.
Graph the solution on the number line.	
Write the solution in interval notation.	There is no solution.

Note:
Exercise:

Problem:

Solve the inequality $\frac{1}{4}x - \frac{1}{12}x > \frac{1}{6}x + \frac{7}{8}$, graph the solution on the number line, and write the solution in interval notation.

Solution:

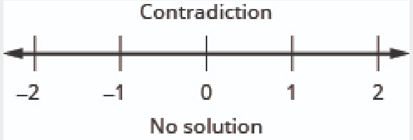


Note:
Exercise:

Problem:

Solve the inequality $\frac{2}{5}z - \frac{1}{3}z < \frac{1}{15}z - \frac{3}{5}$, graph the solution on the number line, and write the solution in interval notation.

Solution:



Translate to an Inequality and Solve

To translate English sentences into inequalities, we need to recognize the phrases that indicate the inequality. Some words are easy, like ‘more than’ and ‘less than’. But others are not as obvious.

Think about the phrase ‘at least’ – what does it mean to be ‘at least 21 years old’? It means 21 or more. The phrase ‘at least’ is the same as ‘greater than or equal to’.

[link] shows some common phrases that indicate inequalities.

$>$	$\geq$	$<$	$\leq$
is greater than	is greater than or equal to	is less than	is less than or equal to

$>$	$\geq$	$<$	$\leq$
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

Example:
Exercise:

Problem:

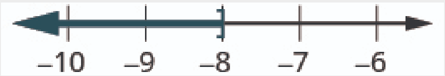
Translate and solve. Then write the solution in interval notation and graph on the number line.

Twelve times *c* is no more than 96.

Solution:
Solution

Translate.	$\text{Twelve times } c \underbrace{\text{ is no more than }}_{} 96$ $12c \leq 96$
Solve—divide both sides by 12.	$\frac{12c}{12} \leq \frac{96}{12}$
Simplify.	$c \leq 8$
Write in interval notation.	$(-\infty, -8]$

Graph on the number line.



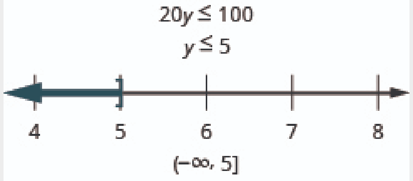
Note:
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Twenty times *y* is at most 100

Solution:


20y ≤ 100
y ≤ 5
4
5
6
7
8
(−∞, 5]

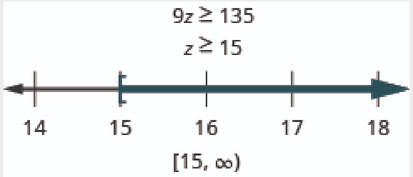
Note:
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Nine times *z* is no less than 135

Solution:


9z ≥ 135
z ≥ 15
14
15
16
17
18
[15, ∞)

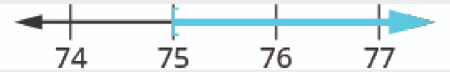
Example:
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Thirty less than *x* is at least 45.

Solution:
Solution

Translate.	Thirty less than x $\underbrace{\text{is at least}}$ 45. $x - 30 \geq 45$
Solve—add 30 to both sides.	$x - 30 + 30 \geq 45 + 30$
Simplify.	$x \geq 75$
Write in interval notation.	$[75, \infty)$
Graph on the number line.	

Note:

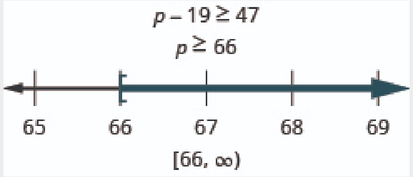
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Nineteen less than *p* is no less than 47

Solution:



$p - 19 \geq 47$
$p \geq 66$
65
66
67
68
69
$[66, \infty)$

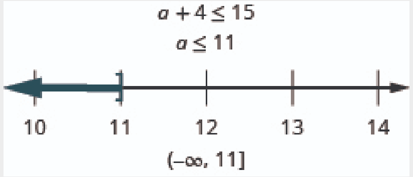
Note:
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Four more than *a* is at most 15.

Solution:



$a + 4 \leq 15$
$a \leq 11$
10
11
12
13
14
$(-\infty, 11]$

Key Concepts

- **Subtraction Property of Inequality**
  For any numbers a, b, and c,
  if $a < b$ then $a - c < b - c$ and
  if $a > b$ then $a - c > b - c$.

- **Addition Property of Inequality**
  For any numbers a, b, and c,
  if $a < b$ then $a + c < b + c$ and
  if $a > b$ then $a + c > b + c$.
- **Division and Multiplication Properties of Inequality**
  For any numbers a, b, and c,
  if $a < b$ and $c > 0$, then $\frac{a}{c} < \frac{b}{c}$ and $ac > bc$.
  if $a > b$ and $c > 0$, then $\frac{a}{c} > \frac{b}{c}$ and $ac > bc$.
  if $a < b$ and $c < 0$, then $\frac{a}{c} > \frac{b}{c}$ and $ac > bc$.
  if $a > b$ and $c < 0$, then $\frac{a}{c} < \frac{b}{c}$ and $ac < bc$.
- When we **divide or multiply** an inequality by a:
  - **positive** number, the inequality stays the **same**.
  - **negative** number, the inequality **reverses**.

Section Exercises

Practice Makes Perfect

Graph Inequalities on the Number Line

In the following exercises, graph each inequality on the number line.
Exercise:

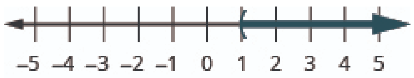
ⓐ $x \le 2$
ⓑ $x > -1$
Problem: ⓒ $x < 0$

Exercise:

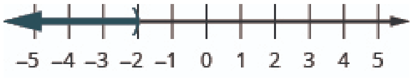
ⓐ $x > 1$
ⓑ $x < -2$
Problem: ⓒ $x \ge -3$

Solution:

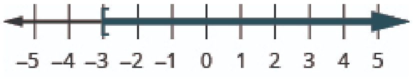
ⓐ



ⓑ



ⓒ



Exercise:

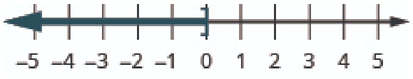
ⓐ $x \geq -3$
ⓑ $x < 4$
Problem: ⓒ $x \leq -2$

Exercise:

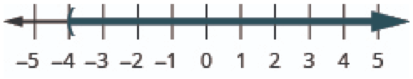
ⓐ $x \leq 0$
ⓑ $x > -4$
Problem: ⓒ $x \geq -1$

Solution:

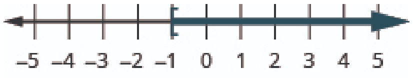
ⓐ



ⓑ



ⓒ



In the following exercises, graph each inequality on the number line and write in interval notation.

Exercise:

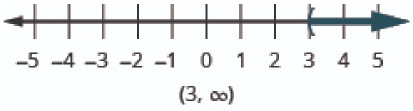
ⓐ $x < -2$
ⓑ $x \geq -3.5$
Problem: ⓒ $x \leq \frac{2}{3}$

Exercise:

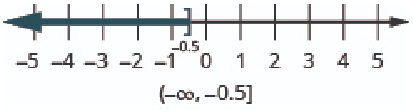
ⓐ $x > 3$
ⓑ $x \leq -0.5$
Problem: ⓒ $x \geq \frac{1}{3}$

Solution:

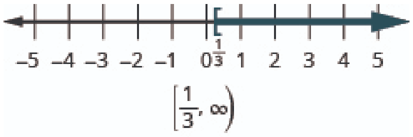
ⓐ


–5 –4 –3 –2 –1 0 1 2 3 4 5
(3, ∞)

ⓑ


–0.5
–5 –4 –3 –2 –1 0 1 2 3 4 5
(–∞, –0.5]

ⓒ


–5 –4 –3 –2 –1 0 1/3 1 2 3 4 5
[1/3, ∞)

Exercise:

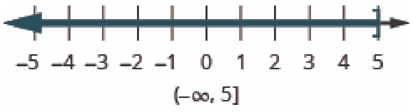
ⓐ $x \geq -4$
ⓑ $x < 2.5$
Problem: ⓒ $x > -\frac{3}{2}$

Exercise:

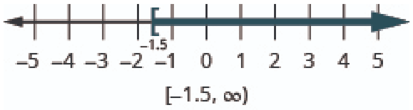
ⓐ $x \leq 5$
ⓑ $x \geq -1.5$
Problem: ⓒ $x < -\frac{7}{3}$

Solution:

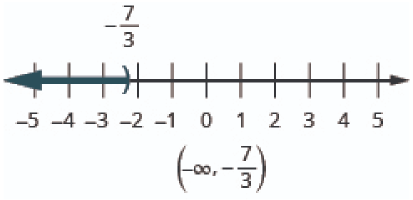
ⓐ


–5 –4 –3 –2 –1 0 1 2 3 4 5
(–∞, 5]

ⓑ


–1.5
–5 –4 –3 –2 –1 0 1 2 3 4 5
[–1.5, ∞)

ⓒ


$-\frac{7}{3}$
–5 –4 –3 –2 –1 0 1 2 3 4 5
$\left(-\infty, -\frac{7}{3}\right)$

Solve Inequalities using the Subtraction and Addition Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

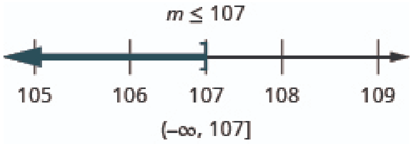
Exercise:

Problem: $n - 11 < 33$

Exercise:

Problem: $m - 45 \leq 62$

Solution:


$m \leq 107$
105
106
107
108
109
$(-\infty, 107]$

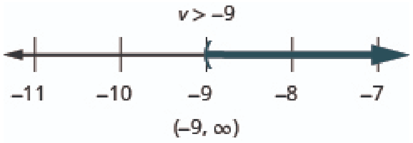
Exercise:

Problem: $u + 25 > 21$

Exercise:

Problem: $v + 12 > 3$

Solution:


$v > -9$
–11
–10
–9
–8
–7
$(-9, \infty)$

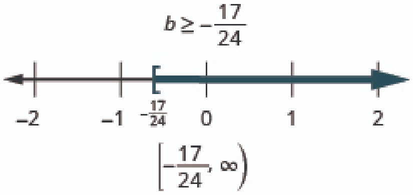
Exercise:

Problem: $a + \frac{3}{4} \geq \frac{7}{10}$

Exercise:

Problem: $b + \frac{7}{8} \geq \frac{1}{6}$

Solution:


$b \geq -\frac{17}{24}$
–2
–1
$-\frac{17}{24}$
0
1
2
$\left[-\frac{17}{24}, \infty\right)$

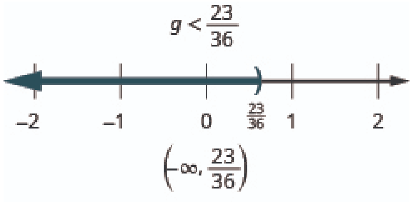
Exercise:

Problem: $f - \frac{13}{20} < -\frac{5}{12}$

Exercise:

Problem: $g - \frac{11}{12} < -\frac{5}{18}$

Solution:


$g < \frac{23}{36}$
–2
–1
0
$\frac{23}{36}$
1
2
$\left(-\infty, \frac{23}{36}\right)$

Solve Inequalities using the Division and Multiplication Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

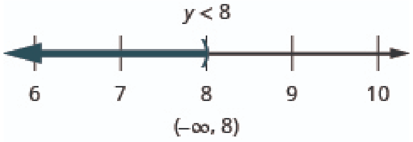
Exercise:

Problem: $8x > 72$

Exercise:

Problem: $6y < 48$

Solution:


$y < 8$
6
7
8
9
10
$(-\infty, 8)$

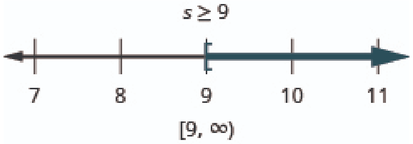
Exercise:

Problem: $7r \leq 56$

Exercise:

Problem: $9s \geq 81$

Solution:


$s \geq 9$
7
8
9
10
11
$[9, \infty)$

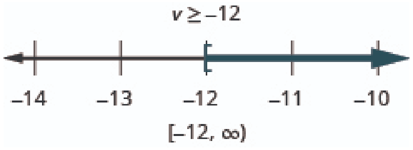
Exercise:

Problem: $-5u \geq 65$

Exercise:

Problem: $-8v \leq 96$

Solution:


v ≥ –12
–14
–13
–12
–11
–10
[–12, ∞)

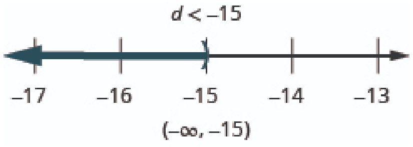
Exercise:

Problem: $-9c < 126$

Exercise:

Problem: $-7d > 105$

Solution:


d < –15
–17
–16
–15
–14
–13
(–∞, –15)

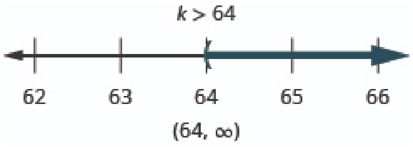
Exercise:

Problem: $20 > \frac{2}{5}h$

Exercise:

Problem: $40 < \frac{5}{8}k$

Solution:


k > 64
62
63
64
65
66
(64, ∞)

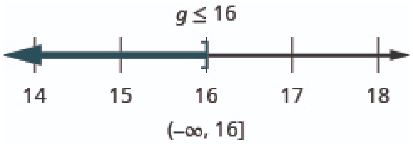
Exercise:

Problem: $\frac{7}{6}j \geq 42$

Exercise:

Problem: $\frac{9}{4}g \leq 36$

Solution:


g ≤ 16
14
15
16
17
18
(−∞, 16]

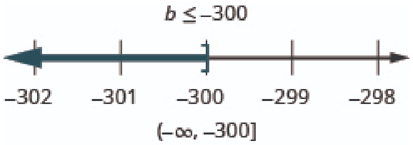
Exercise:

Problem: $\frac{a}{-3} \leq 9$

Exercise:

Problem: $\frac{b}{-10} \geq 30$

Solution:


b ≤ −300
−302
−301
−300
−299
−298
(−∞, −300]

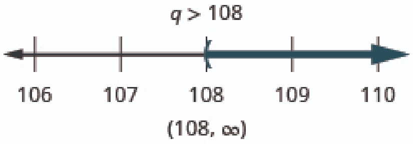
Exercise:

Problem: $-25 < \frac{p}{-5}$

Exercise:

Problem: $-18 > \frac{q}{-6}$

Solution:


q > 108
106
107
108
109
110
(108, ∞)

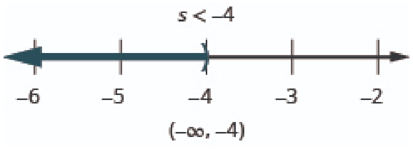
Exercise:

Problem: $9t \geq -27$

Exercise:

Problem: $7s < -28$

Solution:



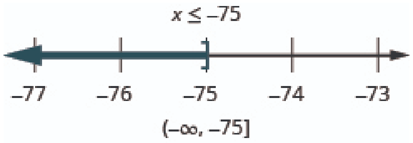
Exercise:

Problem: $\frac{2}{3}y > -36$

Exercise:

Problem: $\frac{3}{5}x \leq -45$

Solution:



Solve Inequalities That Require Simplification

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

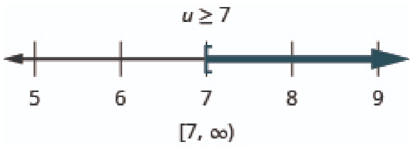
Exercise:

Problem: $4v \geq 9v - 40$

Exercise:

Problem: $5u \leq 8u - 21$

Solution:



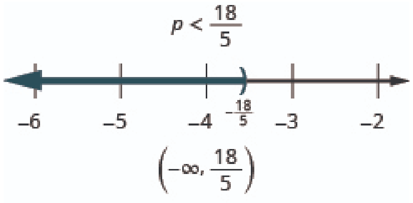
Exercise:

Problem: $13q < 7q - 29$

Exercise:

Problem: $9p > 14p - 18$

Solution:


p < 18/5
−6
−5
−4
−18/5
−3
−2
(−∞, 18/5)

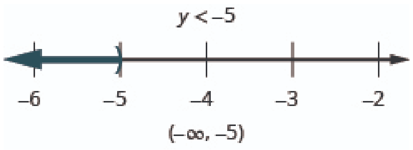
Exercise:

Problem: $12x + 3(x + 7) > 10x - 24$

Exercise:

Problem: $9y + 5(y + 3) < 4y - 35$

Solution:



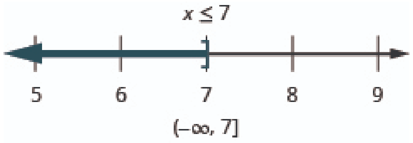
Exercise:

Problem: $6h - 4(h - 1) \leq 7h - 11$

Exercise:

Problem: $4k - (k - 2) \geq 7k - 26$

Solution:


$x \leq 7$
5
6
7
8
9
$(-\infty, 7]$

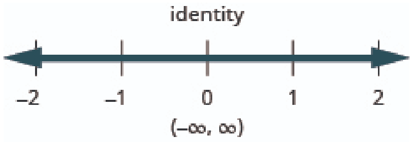
Exercise:

Problem: $8m - 2\left(14 - m\right) \geq 7\left(m - 4\right) + 3m$

Exercise:

Problem: $6n - 12\left(3 - n\right) \leq 9\left(n - 4\right) + 9n$

Solution:


identity
–2
–1
0
1
2
(–∞, ∞)

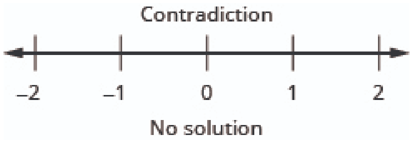
Exercise:

Problem: $\frac{3}{4}b - \frac{1}{3}b < \frac{5}{12}b - \frac{1}{2}$

Exercise:

Problem: $9u + 5(2u - 5) \geq 12(u - 1) + 7u$

Solution:


Contradiction
–2
–1
0
1
2
No solution

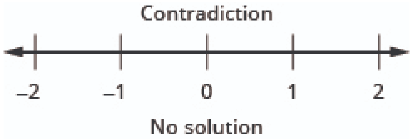
Exercise:

Problem: $\frac{2}{3}g - \frac{1}{2}(g - 14) \leq \frac{1}{6}(g + 42)$

Exercise:

Problem: $\frac{5}{6}a - \frac{1}{4}a > \frac{7}{12}a + \frac{2}{3}$

Solution:


Contradiction
–2
–1
0
1
2
No solution

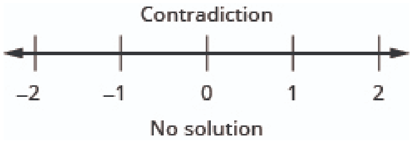
Exercise:

Problem: $\frac{4}{5}h - \frac{2}{3}(h-9) \geq \frac{1}{15}(2h+90)$

Exercise:

Problem: $12v + 3(4v-1) \leq 19(v-2) + 5v$

Solution:


Contradiction
–2
–1
0
1
2
No solution

Mixed practice

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

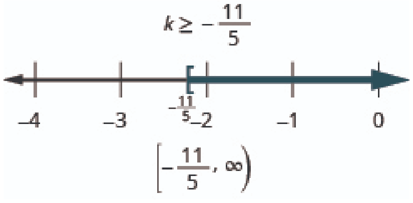
Exercise:

Problem: $15k \leq -40$

Exercise:

Problem: $35k \geq -77$

Solution:


$k \geq -\frac{11}{5}$
–4
–3
$-\frac{11}{5}$
–2
–1
0
$\left[-\frac{11}{5}, \infty\right)$

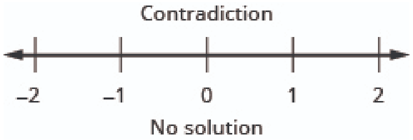
Exercise:

Problem: $23p - 2(6 - 5p) > 3(11p - 4)$

Exercise:

Problem: $18q - 4(10 - 3q) < 5(6q - 8)$

Solution:


Contradiction
–2
–1
0
1
2
No solution

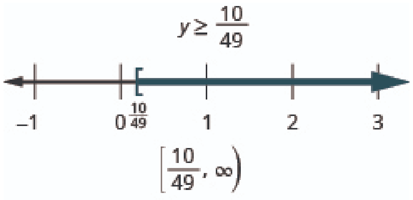
Exercise:

Problem: $-\frac{9}{4}x \geq -\frac{5}{12}$

Exercise:

Problem: $-\frac{21}{8}y \leq -\frac{15}{28}$

Solution:


y ≥ 10/49
–1
0
10/49
1
2
3
[10/49, ∞)

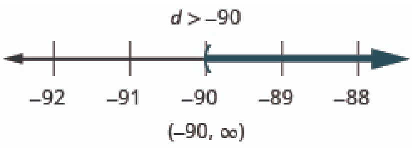
Exercise:

Problem: $c + 34 < -99$

Exercise:

Problem: $d + 29 > -61$

Solution:


d > –90
–92
–91
–90
–89
–88
(–90, ∞)

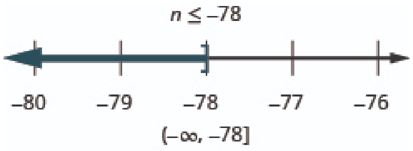
Exercise:

Problem: $\frac{m}{18} \geq -4$

Exercise:

Problem: $\frac{n}{13} \leq -6$

Solution:



Translate to an Inequality and Solve

In the following exercises, translate and solve .Then write the solution in interval notation and graph on the number line.

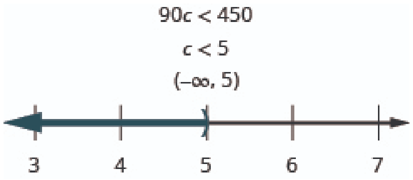
Exercise:

Problem: Fourteen times *d* is greater than 56.

Exercise:

Problem: Ninety times *c* is less than 450.

Solution:



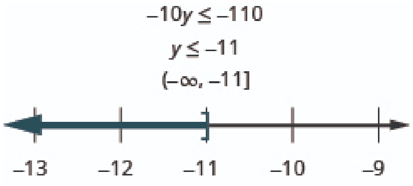
Exercise:

Problem: Eight times *z* is smaller than -40.

Exercise:

Problem: Ten times *y* is at most -110.

Solution:



$-10y \leq -110$

$y \leq -11$

$(-\infty, -11]$

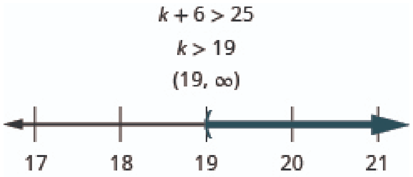
Exercise:

Problem: Three more than *h* is no less than 25.

Exercise:

Problem: Six more than *k* exceeds 25.

Solution:



$k + 6 > 25$

$k > 19$

$(19, \infty)$

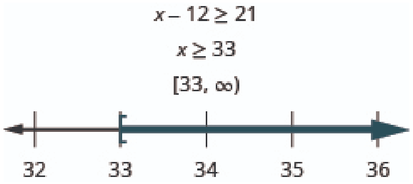
Exercise:

Problem: Ten less than *w* is at least 39.

Exercise:

Problem: Twelve less than *x* is no less than 21.

Solution:



$x - 12 \geq 21$

$x \geq 33$

$[33, \infty)$

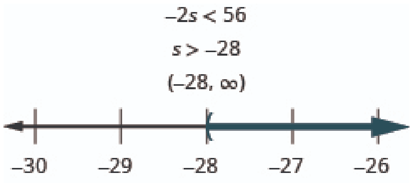
Exercise:

Problem: Negative five times *r* is no more than 95.

Exercise:

Problem: Negative two times *s* is lower than 56.

Solution:



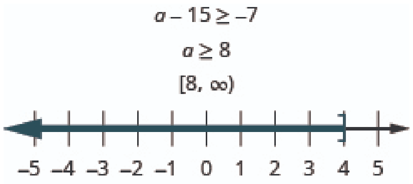
Exercise:

Problem: Nineteen less than *b* is at most -22.

Exercise:

Problem: Fifteen less than *a* is at least -7.

Solution:



Everyday Math

Exercise:

Problem:

Safety A child’s height, *h*, must be at least 57 inches for the child to safely ride in the front seat of a car. Write this as an inequality.

Exercise:

Problem:

Fighter pilots The maximum height, *h*, of a fighter pilot is 77 inches. Write this as an inequality.

Solution:

$h \leq 77$

Exercise:

Problem:

Elevators The total weight, *w*, of an elevator's passengers can be no more than 1,200 pounds. Write this as an inequality.

Exercise:

Problem:

Shopping The number of items, *n*, a shopper can have in the express check-out lane is at most 8. Write this as an inequality.

Solution:

$n \leq 8$

Writing Exercises

Exercise:

Problem: Give an example from your life using the phrase 'at least'.

Exercise:

Problem: Give an example from your life using the phrase 'at most'.

Solution:

Answers will vary.

Exercise:

Problem:

Explain why it is necessary to reverse the inequality when solving $-5x > 10$.

Exercise:

Problem:

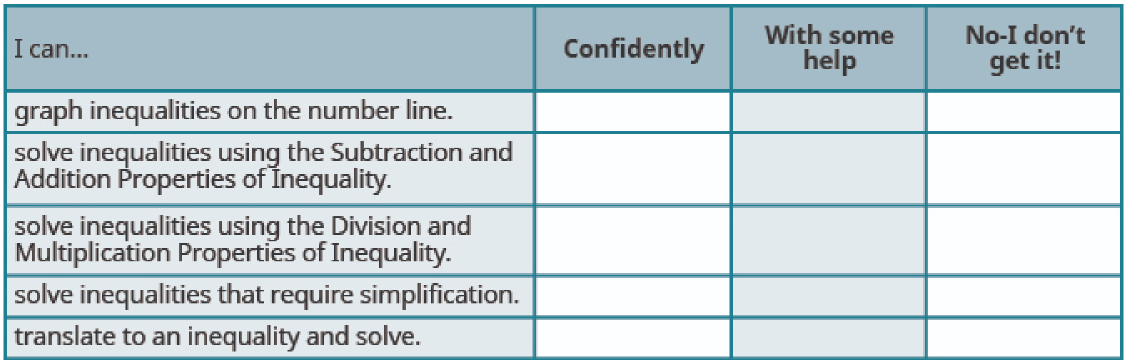
Explain why it is necessary to reverse the inequality when solving $\frac{n}{-3} < 12$.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
graph inequalities on the number line.			
solve inequalities using the Subtraction and Addition Properties of Inequality.			
solve inequalities using the Division and Multiplication Properties of Inequality.			
solve inequalities that require simplification.			
translate to an inequality and solve.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Chapter 2 Review Exercises

Section 2.1 Solve Equations using the Subtraction and Addition Properties of Equality

Verify a Solution of an Equation

In the following exercises, determine whether each number is a solution to the equation.
Exercise:

Problem: $10x - 1 = 5x; x = \frac{1}{5}$

Exercise:

Problem: $w + 2 = \frac{5}{8}; w = \frac{3}{8}$

Solution:

no

Exercise:

Problem: $-12n + 5 = 8n; n = -\frac{5}{4}$

Exercise:

Problem: $6a - 3 = -7a, a = \frac{3}{13}$

Solution:

yes

Solve Equations using the Subtraction and Addition Properties of Equality

In the following exercises, solve each equation using the Subtraction Property of Equality.

Exercise:

Problem: $x + 7 = 19$

Exercise:

Problem: $y + 2 = -6$

Solution:

$y = -8$

Exercise:

Problem: $a + \frac{1}{3} = \frac{5}{3}$

Exercise:

Problem: $n + 3.6 = 5.1$

Solution:

$n = 1.5$

In the following exercises, solve each equation using the Addition Property of Equality.

Exercise:

Problem: $u - 7 = 10$

Exercise:

Problem: $x - 9 = -4$

Solution:

$x = 5$

Exercise:

Problem: $c - \frac{3}{11} = \frac{9}{11}$

Exercise:

Problem: $p - 4.8 = 14$

Solution:

$p = 18.8$

In the following exercises, solve each equation.

Exercise:

Problem: $n - 12 = 32$

Exercise:

Problem: $y + 16 = -9$

Solution:

$y = -25$

Exercise:

Problem: $f + \frac{2}{3} = 4$

Exercise:

Problem: $d - 3.9 = 8.2$

Solution:

$d = 12.1$

Solve Equations That Require Simplification

In the following exercises, solve each equation.

Exercise:

Problem: $y+8-15=-3$

Exercise:

Problem: $7x+10-6x+3=5$

Solution:

$x=-8$

Exercise:

Problem: $6(n-1)-5n=-14$

Exercise:

Problem: $8(3p+5)-23(p-1)=35$

Solution:

$p=-28$

Translate to an Equation and Solve

In the following exercises, translate each English sentence into an algebraic equation and then solve it.

Exercise:

Problem: The sum of -6 and m is 25.

Exercise:

Problem: Four less than n is 13.

Solution:

$n-4=13; n=17$

Translate and Solve Applications

In the following exercises, translate into an algebraic equation and solve.

Exercise:

Problem:

Rochelle’s daughter is 11 years old. Her son is 3 years younger. How old is her son?

Exercise:

Problem:

Tan weighs 146 pounds. Minh weighs 15 pounds more than Tan. How much does Minh weigh?

Solution:

161 pounds

Exercise:

Problem:

Peter paid $9.75 to go to the movies, which was $46.25 less than he paid to go to a concert. How much did he pay for the concert?

Exercise:

Problem:

Elissa earned $152.84 this week, which was $21.65 more than she earned last week. How much did she earn last week?

Solution:

$131.19

Section 2.2 Solve Equations using the Division and Multiplication Properties of Equality

Solve Equations Using the Division and Multiplication Properties of Equality

In the following exercises, solve each equation using the division and multiplication properties of equality and check the solution.

Exercise:

Problem: $8x = 72$

Exercise:

Problem: $13a = -65$

Solution:

$a = -5$

Exercise:

Problem: $0.25p = 5.25$

Exercise:

Problem: $-y = 4$

Solution:

$y = -4$

Exercise:

Problem: $\frac{n}{6} = 18$

Exercise:

Problem: $\frac{y}{-10} = 30$

Solution:

$y = -300$

Exercise:

Problem: $36 = \frac{3}{4}x$

Exercise:

Problem: $\frac{5}{8}u = \frac{15}{16}$

Solution:

$u = \frac{3}{2}$

Exercise:

Problem: $-18m = -72$

Exercise:

Problem: $\frac{c}{9} = 36$

Solution:

$c = 324$

Exercise:

Problem: $0.45x = 6.75$

Exercise:

Problem: $\frac{11}{12} = \frac{2}{3}y$

Solution:

$y = \frac{11}{8}$

Solve Equations That Require Simplification

In the following exercises, solve each equation requiring simplification.

Exercise:

Problem: $5r - 3r + 9r = 35 - 2$

Exercise:

Problem: $24x + 8x - 11x = -7 - 14$

Solution:

$x = -1$

Exercise:

Problem: $\frac{11}{12}n - \frac{5}{6}n = 9 - 5$

Exercise:

Problem: $-9(d - 2) - 15 = -24$

Solution:

$d = 3$

Translate to an Equation and Solve

In the following exercises, translate to an equation and then solve.

Exercise:

Problem: 143 is the product of -11 and y.

Exercise:

Problem: The quotient of b and and 9 is -27.

Solution:

$\frac{b}{9} = -27; b = -243$

Exercise:

Problem: The sum of q and one-fourth is one.

Exercise:

Problem: The difference of s and one-twelfth is one fourth.

Solution:

$s - \frac{1}{12} = \frac{1}{4}; s = \frac{1}{3}$

Translate and Solve Applications

In the following exercises, translate into an equation and solve.

Exercise:

Problem:

Ray paid $21 for 12 tickets at the county fair. What was the price of each ticket?

Exercise:

Problem:

Janet gets paid $24 per hour. She heard that this is $\frac{3}{4}$ of what Adam is paid. How much is Adam paid per hour?

Solution:

$32

Section 2.3 Solve Equations with Variables and Constants on Both Sides

Solve an Equation with Constants on Both Sides

In the following exercises, solve the following equations with constants on both sides.
Exercise:

Problem: $8p + 7 = 47$

Exercise:

Problem: $10w - 5 = 65$

Solution:

$w = 7$

Exercise:

Problem: $3x + 19 = -47$

Exercise:

Problem: $32 = -4 - 9n$

Solution:

$n = -4$

Solve an Equation with Variables on Both Sides

In the following exercises, solve the following equations with variables on both sides.
Exercise:

Problem: $7y = 6y - 13$

Exercise:

Problem: $5a + 21 = 2a$

Solution:

$a = -7$

Exercise:

Problem: $k = -6k - 35$

Exercise:

Problem: $4x - \frac{3}{8} = 3x$

Solution:

$x = \frac{3}{8}$

Solve an Equation with Variables and Constants on Both Sides

In the following exercises, solve the following equations with variables and constants on both sides.

Exercise:

Problem: $12x - 9 = 3x + 45$

Exercise:

Problem: $5n - 20 = -7n - 80$

Solution:

$n = -5$

Exercise:

Problem: $4u + 16 = -19 - u$

Exercise:

Problem: $\frac{5}{8}c - 4 = \frac{3}{8}c + 4$

Solution:

$c = 32$

Section 2.4 Use a General Strategy for Solving Linear Equations

Solve Equations Using the General Strategy for Solving Linear Equations

In the following exercises, solve each linear equation.

Exercise:

Problem: $6(x+6)=24$

Exercise:

Problem: $9(2p-5)=72$

Solution:

$p=\frac{13}{2}$

Exercise:

Problem: $-(s+4)=18$

Exercise:

Problem: $8+3(n-9)=17$

Solution:

$n=12$

Exercise:

Problem: $23-3(y-7)=8$

Exercise:

Problem: $\frac{1}{3}(6m+21)=m-7$

Solution:

$m=-14$

Exercise:

Problem: $4(3.5y+0.25)=365$

Exercise:

Problem: $0.25(q-8)=0.1(q+7)$

Solution:

$q = 18$

Exercise:

Problem: $8(r - 2) = 6(r + 10)$

Exercise:

Problem: $5 + 7(2 - 5x) = 2(9x + 1) -(13x - 57)$

Solution:

$x = -1$

Exercise:

Problem: $(9n + 5) - (3n - 7) = 20 - (4n - 2)$

Exercise:

Problem: $2[-16 + 5(8k - 6)] = 8(3 - 4k) - 32$

Solution:

$k = \frac{3}{4}$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

Exercise:

Problem: $17y - 3(4 - 2y) = 11(y - 1) +12y - 1$

Exercise:

Problem: $9u + 32 = 15(u - 4) -3(2u + 21)$

Solution:

contradiction; no solution

Exercise:

Problem: $-8(7m+4)=-6(8m+9)$

Exercise:

Problem: $21(c-1)-19(c+1)=2(c-20)$

Solution:

identity; all real numbers

Section 2.5 Solve Equations with Fractions and Decimals

Solve Equations with Fraction Coefficients

In the following exercises, solve each equation with fraction coefficients.

Exercise:

Problem: $\frac{2}{5}n-\frac{1}{10}=\frac{7}{10}$

Exercise:

Problem: $\frac{1}{3}x+\frac{1}{5}x=8$

Solution:

$x=15$

Exercise:

Problem: $\frac{3}{4}a-\frac{1}{3}=\frac{1}{2}a-\frac{5}{6}$

Exercise:

Problem: $\frac{1}{2}(k-3)=\frac{1}{3}(k+16)$

Solution:

$k = 41$

Exercise:

Problem: $\frac{3x-2}{5} = \frac{3x+4}{8}$

Exercise:

Problem: $\frac{5y-1}{3} + 4 = \frac{-8y+4}{6}$

Solution:

$y = -1$

Solve Equations with Decimal Coefficients

In the following exercises, solve each equation with decimal coefficients.

Exercise:

Problem: $0.8x - 0.3 = 0.7x + 0.2$

Exercise:

Problem: $0.36u + 2.55 = 0.41u + 6.8$

Solution:

$u = -85$

Exercise:

Problem: $0.6p - 1.9 = 0.78p + 1.7$

Exercise:

Problem: $0.6p - 1.9 = 0.78p + 1.7$

Solution:

$d = -20$

Section 2.6 Solve a Formula for a Specific Variable

Use the Distance, Rate, and Time Formula

In the following exercises, solve.

Exercise:

Problem:

Natalie drove for $7\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?

Exercise:

Problem:

Mallory is taking the bus from St. Louis to Chicago. The distance is 300 miles and the bus travels at a steady rate of 60 miles per hour. How long will the bus ride be?

Solution:

5 hours

Exercise:

Problem:

Aaron's friend drove him from Buffalo to Cleveland. The distance is 187 miles and the trip took 2.75 hours. How fast was Aaron's friend driving?

Exercise:

Problem:

Link rode his bike at a steady rate of 15 miles per hour for $2\frac{1}{2}$ hours. How much distance did he travel?

Solution:

37.5 miles

Solve a Formula for a Specific Variable

In the following exercises, solve.

Exercise:

Problem: Use the formula. $d=rt$ to solve for t
ⓐ when $d=510$ and $r=60$
ⓑ in general

Exercise:

Problem: Use the formula. $d = rt$ to solve for r
ⓐ when when $d = 451$ and $t = 5.5$
ⓑ in general

Solution:

ⓐ $r = 82\,\text{mph}$; ⓑ $r = \frac{D}{t}$

Exercise:

Problem: Use the formula $A = \frac{1}{2}bh$ to solve for b
ⓐ when $A = 390$ and $h = 26$
ⓑ in general

Exercise:

Problem: Use the formula $A = \frac{1}{2}bh$ to solve for h
ⓐ when $A = 153$ and $b = 18$
ⓑ in general

Solution:

ⓐ $h = 17$ ⓑ $h = \frac{2A}{b}$

Exercise:

Problem: Use the formula $I = Prt$ to solve for the principal, P for
ⓐ $I = \$2,501, r = 4.1\%$,
$t = 5\,\text{years}$
ⓑ in general

Exercise:

Problem: Solve the formula $4x + 3y = 6$ for y
ⓐ when $x = -2$
ⓑ in general

Solution:

ⓐ $y = \frac{14}{3}$ ⓑ $y = \frac{6-4x}{3}$

Exercise:

Problem: Solve $180 = a + b + c$ for c.

Exercise:

Problem: Solve the formula $V = LWH$ for H.

Solution:

$H = \frac{V}{LW}$

Section 2.7 Solve Linear Inequalities

Graph Inequalities on the Number Line

In the following exercises, graph each inequality on the number line.

Exercise:

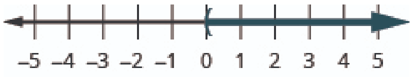
ⓐ $x \le 4$
ⓑ $x > -2$
Problem: ⓒ $x < 1$

Exercise:

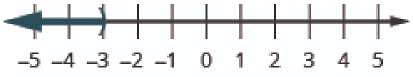
ⓐ $x > 0$
ⓑ $x < -3$
Problem: ⓒ $x \ge -1$

Solution:

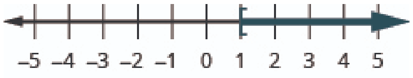
ⓐ


–5 –4 –3 –2 –1 0 1 2 3 4 5

ⓑ


–5 –4 –3 –2 –1 0 1 2 3 4 5

ⓒ


–5 –4 –3 –2 –1 0 1 2 3 4 5

In the following exercises, graph each inequality on the number line and write in interval notation.

Exercise:

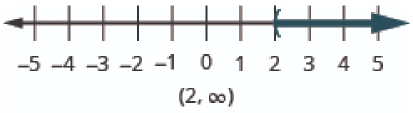
Problem:
ⓐ $x < -1$
ⓑ $x \geq -2.5$
ⓒ $x \leq \frac{5}{4}$

Exercise:

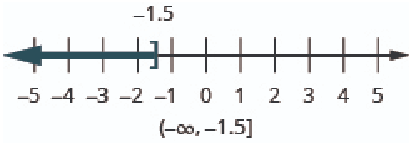
Problem:
ⓐ $x > 2$
ⓑ $x \leq -1.5$
ⓒ $x \geq \frac{5}{3}$

Solution:

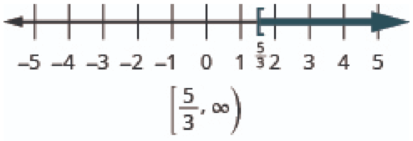
ⓐ


–5 –4 –3 –2 –1 0 1 2 3 4 5
(2, ∞)

ⓑ


–1.5
–5 –4 –3 –2 –1 0 1 2 3 4 5
(–∞, –1.5]

ⓒ


–5 –4 –3 –2 –1 0 1 5/3 2 3 4 5
[5/3, ∞)

Solve Inequalities using the Subtraction and Addition Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

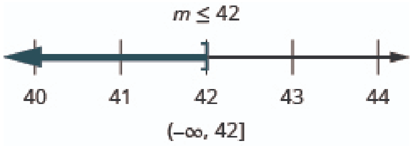
Exercise:

Problem: $n - 12 \leq 23$

Exercise:

Problem: $m + 14 \leq 56$

Solution:



$m \leq 42$
40
41
42
43
44
$(-\infty, 42]$

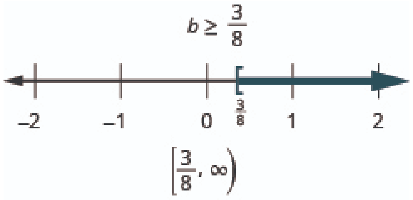
Exercise:

Problem: $a + \frac{2}{3} \geq \frac{7}{12}$

Exercise:

Problem: $b - \frac{7}{8} \geq -\frac{1}{2}$

Solution:



$b \geq \frac{3}{8}$
–2
–1
0
$\frac{3}{8}$
1
2
$\left[\frac{3}{8}, \infty\right)$

Solve Inequalities using the Division and Multiplication Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

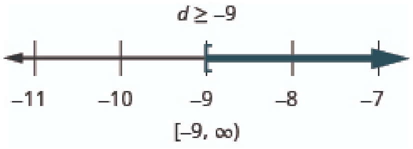
Exercise:

Problem: $9x > 54$

Exercise:

Problem: $-12d \leq 108$

Solution:


$d \geq -9$
–11
–10
–9
–8
–7
[–9, ∞)

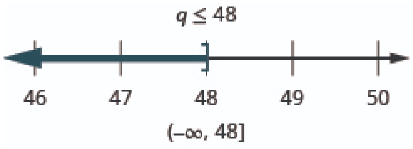
Exercise:

Problem: $\frac{5}{2}j < -60$

Exercise:

Problem: $\frac{q}{-2} \geq -24$

Solution:


$q \leq 48$
46
47
48
49
50
(–∞, 48]

Solve Inequalities That Require Simplification

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

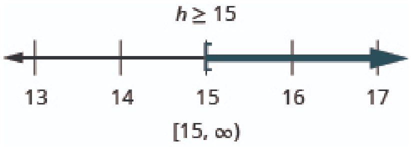
Exercise:

Problem: $6p > 15p - 30$

Exercise:

Problem: $9h - 7(h - 1) \leq 4h - 23$

Solution:


$h \geq 15$
13
14
15
16
17
[15, ∞)

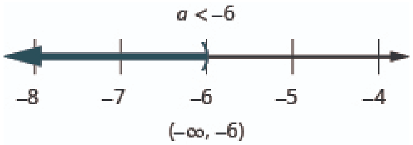
Exercise:

Problem: $5n - 15(4 - n) < 10(n - 6) + 10n$

Exercise:

Problem: $\frac{3}{8}a - \frac{1}{12}a > \frac{5}{12}a + \frac{3}{4}$

Solution:



Translate to an Inequality and Solve

In the following exercises, translate and solve. Then write the solution in interval notation and graph on the number line.

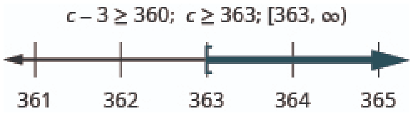
Exercise:

Problem: Five more than *z* is at most 19.

Exercise:

Problem: Three less than *c* is at least 360.

Solution:


c – 3 ≥ 360; c ≥ 363; [363, ∞)
361
362
363
364
365

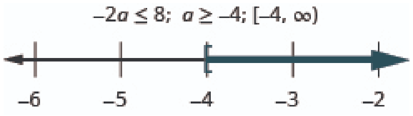
Exercise:

Problem: Nine times *n* exceeds 42.

Exercise:

Problem: Negative two times *a* is no more than 8.

Solution:


–2a ≤ 8; a ≥ –4; [–4, ∞)
–6
–5
–4
–3
–2

Everyday Math

Exercise:

Problem:

Describe how you have used two topics from this chapter in your life outside of your math class during the past month.

Chapter 2 Practice Test

Exercise:

Problem:

Determine whether each number is a solution to the equation $6x - 3 = x + 20$.

ⓐ 5
ⓑ $\frac{23}{5}$

Solution:

ⓐ no ⓑ yes

In the following exercises, solve each equation.

Exercise:

Problem: $n - \frac{2}{3} = \frac{1}{4}$

Exercise:

Problem: $\frac{9}{2}c = 144$

Solution:

$c = 32$

Exercise:

Problem: $4y - 8 = 16$

Exercise:

Problem: $-8x - 15 + 9x - 1 = -21$

Solution:

$x = -5$

Exercise:

Problem: $-15a = 120$

Exercise:

Problem: $\frac{2}{3}x = 6$

Solution:

$x = 9$

Exercise:

Problem: $x - 3.8 = 8.2$

Exercise:

Problem: $10y = -5y - 60$

Solution:

$y = -4$

Exercise:

Problem: $8n - 2 = 6n - 12$

Exercise:

Problem: $9m - 2 - 4m - m = 42 - 8$

Solution:

$m = 9$

Exercise:

Problem: $-5(2x - 1) = 45$

Exercise:

Problem: $-(d-9)=23$

Solution:

$d=-14$

Exercise:

Problem: $\frac{1}{4}(12m-28)=6-2(3m-1)$

Exercise:

Problem: $2(6x-5)-8=-22$

Solution:

$x=-\frac{1}{3}$

Exercise:

Problem: $8(3a-5)-7(4a-3)=20-3a$

Exercise:

Problem: $\frac{1}{4}p-\frac{1}{3}=\frac{1}{2}$

Solution:

$p=\frac{10}{3}$

Exercise:

Problem: $0.1d+0.25(d+8)=4.1$

Exercise:

Problem: $14n-3(4n+5)=-9+2(n-8)$

Solution:

contradiction; no solution

Exercise:

Problem: $9(3u-2)-4[6-8(u-1)]=3(u-2)$

Exercise:

Solve the formula $x-2y=5$ for y
ⓐ when $x=-3$
Problem: ⓑ in general

Solution:

ⓐ $y=4$ ⓑ $y=\frac{5-x}{2}$

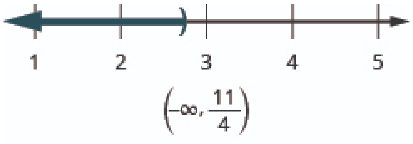
In the following exercises, graph on the number line and write in interval notation.
Exercise:

Problem: $x \geq -3.5$

Exercise:

Problem: $x < \frac{11}{4}$

Solution:



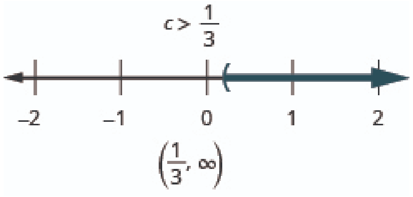
In the following exercises,, solve each inequality, graph the solution on the number line, and write the solution in interval notation.
Exercise:

Problem: $8k \geq 5k-120$

Exercise:

Problem: $3c-10(c-2) < 5c+16$

Solution:



In the following exercises, translate to an equation or inequality and solve.

Exercise:

Problem: 4 less than twice *x* is 16.

Exercise:

Problem: Fifteen more than *n* is at least 48.

Solution:

$n + 15 \geq 48; n \geq 33$

Exercise:

Problem:

Samuel paid $25.82 for gas this week, which was $3.47 less than he paid last week. How much had he paid last week?

Exercise:

Problem:

Jenna bought a coat on sale for $120, which was $\frac{2}{3}$ of the original price. What was the original price of the coat?

Solution:

$120 = \frac{2}{3}p$; The original price was $180.

Exercise:

Problem:

Sean took the bus from Seattle to Boise, a distance of 506 miles. If the trip took $7\frac{2}{3}$ hours, what was the speed of the bus?

Exponential Functions
In this section, you will:

- Evaluate exponential functions.
- Find the equation of an exponential function.
- Use compound interest formulas.
- Evaluate exponential functions with base e.

India is the second most populous country in the world with a population of about 1.25 billion people in 2013. The population is growing at a rate of about 1.2% each year[footnote]. If this rate continues, the population of India will exceed China's population by the year 2031.When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions*, which model this kind of rapid growth.
http://www.worldometers.info/world-population/. Accessed February 24, 2014.

Identifying Exponential Functions

When exploring linear growth, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input. For example, in the equation $f(x)=3x+4$,the slope tells us the output increases by 3 each time the input increases by 1. The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009 to 2011. In 2011, 2.5% of the population was vegan, adhering to a diet that does not include any animal products—no meat, poultry, fish, dairy, or eggs. If this rate continues, vegans will make up 10% of the U.S. population in 2015, 40% in 2019, and 80% in 2021.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of exponential growth, let us contrast exponential growth with linear growth. We will construct two functions. The first function is exponential. We will start with an input of 0, and increase each input by 1. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of 0, and increase each input by 1. We will add 2 to the corresponding consecutive outputs. See [link].

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

From [link] we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** refers to the original value from the range increases by the *same percentage* over equal increments found in the domain.
- **Linear growth** refers to the original value from the range increases by the *same amount* over equal increments found in the domain.

Apparently, the difference between "the same percentage" and "the same amount" is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding 2 to the output whenever the input was increased by one.

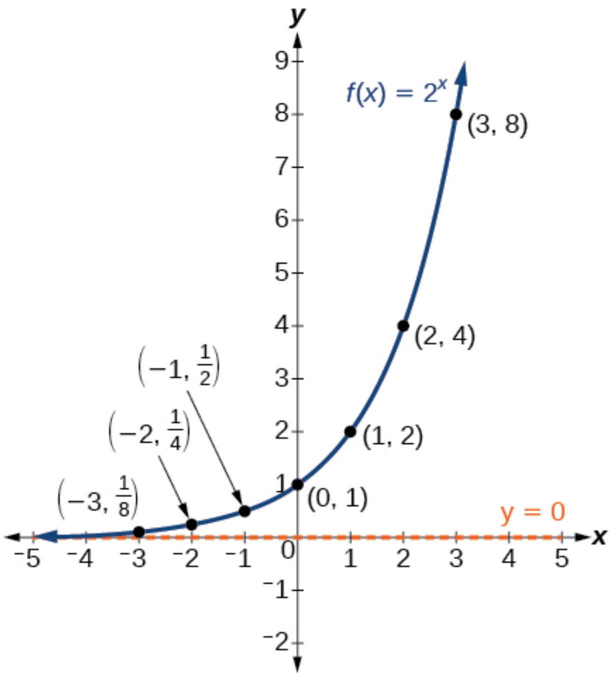
The general form of the exponential function is $f(x) = ab^x,$ where a is any nonzero number, b is a positive real number not equal to 1.

- If $b > 1$,the function grows at a rate proportional to its size.
- If $0 < b < 1$, the function decays at a rate proportional to its size.

Let's look at the function $f(x) = 2^x$ from our example. We will create a table ([link]) to determine the corresponding outputs over an interval in the domain from -3 to 3.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$2^{-3} = \frac{1}{8}$	$2^{-2} = \frac{1}{4}$	$2^{-1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$

Let us examine the graph of f by plotting the ordered pairs we observe on the table in [link], and then make a few observations.



Let’s define the behavior of the graph of the exponential function $f(x) = 2^x$ and highlight some its key characteristics.

- the domain is $(-\infty, \infty)$,
- the range is $(0, \infty)$,
- as $x \to \infty$, $f(x) \to \infty$,
- as $x \to -\infty$, $f(x) \to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the *x*-axis because base two raised to any exponent never has the result of zero.
- $y = 0$ is the horizontal asymptote.
- the *y*-intercept is 1.

Note:
Exponential Function
For any real number x,an exponential function is a function with the form
Equation:

$$f(x) = ab^x$$

where

- a is a non-zero real number called the initial value and
- b is any positive real number such that $b \neq 1$.
- The domain of f is all real numbers.
- The range of f is all positive real numbers if $a > 0$.
- The range of f is all negative real numbers if $a < 0$.
- The *y*-intercept is $(0, a)$,and the horizontal asymptote is $y = 0$.

Example:
Exercise:

Problem:
Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

Solution:

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x) = x^3$ is a power function.

Recall that the base *b* of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

Note:
Exercise:

Problem: Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

Solution:

$g(x) = 0.875^x$ and $j(x) = 1095.6^{-2x}$ represent exponential functions.

Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than 1. Why do we limit the base b to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

- Let $b = -9$ and $x = \frac{1}{2}$. Then $f(x) = f\left(\frac{1}{2}\right) = (-9)^{\frac{1}{2}} = \sqrt{-9}$, which is not a real number.

Why do we limit the base to positive values other than 1? Because base 1 results in the constant function. Observe what happens if the base is 1 :

- Let $b = 1$. Then $f(x) = 1^x = 1$ for any value of x.

To evaluate an exponential function with the form $f(x) = b^x$,we simply substitute x with the given value, and calculate the resulting power. For example:

Let $f(x) = 2^x$. What is $f(3)$?
Equation:

$$\begin{array}{lll} f\left(x\right) & = 2^x & \\ f\left(3\right) & = 2^3 & \text{Substitute } x = 3. \\ & = 8 & \text{Evaluate the power.} \end{array}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let $f(x) = 30(2)^x$. What is $f(3)$?
Equation:

$$\begin{array}{lll} f\left(x\right) & = 30(2)^x & \\ f\left(3\right) & = 30(2)^3 & \text{Substitute } x = 3. \\ & = 30\left(8\right) & \text{Simplify the power first.} \\ & = 240 & \text{Multiply.} \end{array}$$

Note that if the order of operations were not followed, the result would be incorrect:
Equation:

$$f(3) = 30(2)^3 \neq 60^3 = 216{,}000$$

Example:
Exercise:

Problem:
Evaluating Exponential Functions

Let $f\left(x\right) = 5(3)^{x+1}$. Evaluate $f\left(2\right)$ without using a calculator.

Solution:

Follow the order of operations. Be sure to pay attention to the parentheses.
Equation:

$$\begin{array}{lll} f\left(x\right) & = 5(3)^{x+1} & \\ f\left(2\right) & = 5(3)^{2+1} & \text{Substitute } x = 2. \\ & = 5(3)^3 & \text{Add the exponents.} \\ & = 5\left(27\right) & \text{Simplify the power.} \\ & = 135 & \text{Multiply.} \end{array}$$

Note:
Exercise:

Problem: Let $f(x) = 8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

Solution:

5.5556

Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term “exponential growth” is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

Note:
Exponential Growth
A function that models **exponential growth** grows by a rate proportional to the amount present. For any real number x and any positive real numbers a and b such that $b \neq 1$,an exponential growth function has the form
Equation:

$$f(x) = ab^{x}$$

where

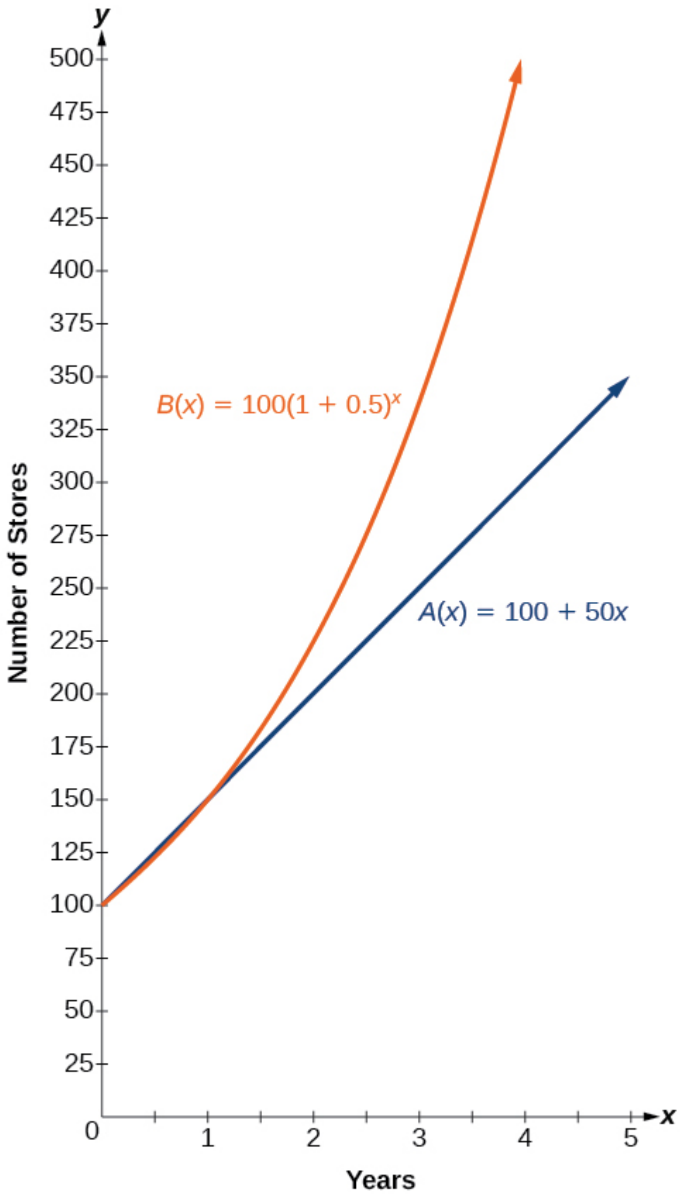
- a is the initial or starting value of the function.
- b is the growth factor or growth multiplier per unit x.

In more general terms, we have an *exponential function*, in which a constant base is raised to a variable exponent. To differentiate between linear and exponential functions, let’s consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function $A(x) = 100 + 50x$. Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function $B(x) = 100(1 + 0.5)^{x}$.

A few years of growth for these companies are illustrated in [link].

Year, x	**Stores, Company A**	**Stores, Company B**
0	$100+50\left(0\right)=100$	$100{\left(1+0.5\right)}^{0}=100$
1	$100+50\left(1\right)=150$	$100{\left(1+0.5\right)}^{1}=150$
2	$100+50\left(2\right)=200$	$100{\left(1+0.5\right)}^{2}=225$
3	$100+50\left(3\right)=250$	$100{\left(1+0.5\right)}^{3}=337.5$
x	$A\left(x\right)=100+50x$	$B\left(x\right)=100{\left(1+0.5\right)}^{x}$

The graphs comparing the number of stores for each company over a five-year period are shown in [link]. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.



The graph shows the numbers of stores Companies A and B opened over a five-year period.

Notice that the domain for both functions is $[0, \infty)$,and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A.

Now we will turn our attention to the function representing the number of stores for Company B, $B(x) = 100(1 + 0.5)^x$. In this exponential function, 100 represents the initial number of stores, 0.50 represents the growth rate, and $1 + 0.5 = 1.5$ represents the growth factor. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$,where 100 is the initial value, 1.5 is called the *base*, and x is called the *exponent*.

Example:
Exercise:

Problem:
Evaluating a Real-World Exponential Model

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about $1.2\%.$ This situation is represented by the growth function $P(t)=1.25(1.012)^t$, where t is the number of years since $2013.$ To the nearest thousandth, what will the population of India be in 2031?

Solution:

To estimate the population in 2031, we evaluate the models for $t=18$, because 2031 is 18 years after 2013. Rounding to the nearest thousandth,
Equation:

$$P(18)=1.25(1.012)^{18}\approx 1.549$$

There will be about 1.549 billion people in India in the year 2031.

Note:
Exercise:

Problem:

The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about $0.6\%.$ This situation is represented by the growth function $P(t)=1.39(1.006)^t$, where t is the number of years since $2013.$To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in [link]?

Solution:

About 1.548 billion people; by the year 2031, India's population will exceed China's by about 0.001 billion, or 1 million people.

Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants a and b,and evaluate the function.

Note:
Given two data points, write an exponential model.

1. If one of the data points has the form $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
3. Using the a and b found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

Example:
Exercise:

Problem:
Writing an Exponential Model When the Initial Value Is Known

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an algebraic function $N(t)$ representing the population (N) of deer over time t.

Solution:

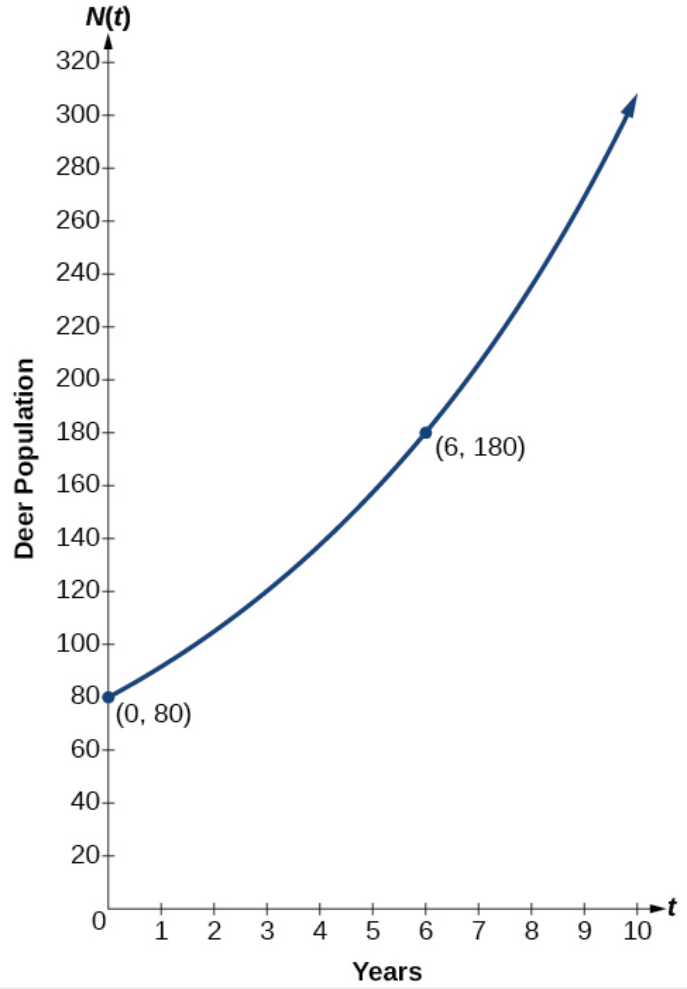
We let our independent variable t be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: (0, 80) and (6, 180). Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, $a = 80$. We can now substitute the second point into the equation $N(t) = 80b^t$ to find b :
Equation:

$$\begin{aligned}
N(t) &= 80b^t && \\
180 &= 80b^6 && \text{Substitute using point } (6,\ 180). \\
\tfrac{9}{4} &= b^6 && \text{Divide and write in lowest terms.} \\
b &= \left(\tfrac{9}{4}\right)^{\frac{1}{6}} && \text{Isolate } b \text{ using properties of exponents.} \\
b &\approx 1.1447 && \text{Round to 4 decimal places.}
\end{aligned}$$

NOTE: *Unless otherwise stated, do not round any intermediate calculations. Then round the final answer to four places for the remainder of this section.*

The exponential model for the population of deer is $N(t) = 80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs gets large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in [link] passes through the initial points given in the problem, $(0,\ 80)$ and $(6,\ 180)$. We can also see that the domain for the function is $[0, \infty)$,and the range for the function is $[80, \infty)$.



Graph showing the population of deer over time, $N(t) = 80(1.1447)^t$, t years after 2006

Note:
Exercise:

Problem:

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

Solution:

$(0, 129)$ and $(2, 236)$; $N(t) = 129(1.3526)^t$

Example:
Exercise:

Problem:
Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$.

Solution:

Because we don’t have the initial value, we substitute both points into an equation of the form $f(x) = ab^x$, and then solve the system for a and b.

- Substituting $(-2, 6)$ gives $6 = ab^{-2}$
- Substituting $(2, 1)$ gives $1 = ab^2$

Use the first equation to solve for a in terms of b :

$$\begin{array}{ll} 6 = ab^{-2} & \\ \dfrac{6}{b^{-2}} = a & \text{Divide.} \\ a = 6b^2 & \text{Use properties of exponents to rewrite the denominator.} \end{array}$$

Substitute a in the second equation, and solve for b :

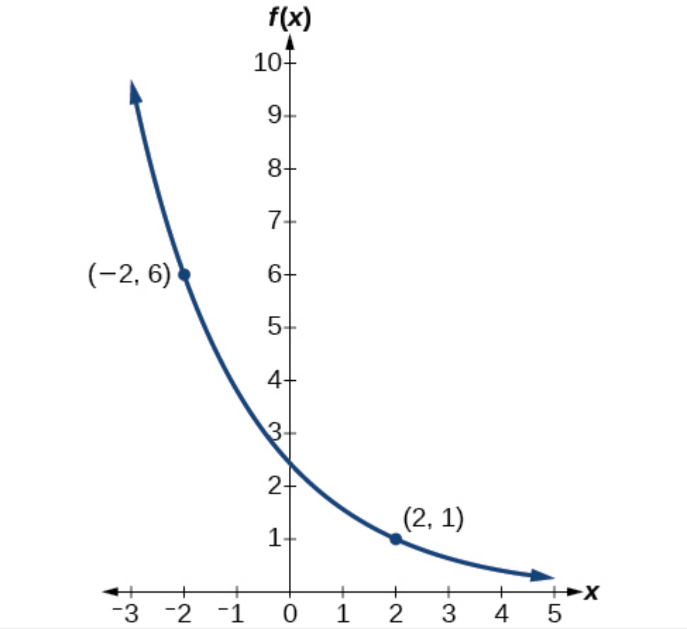
$$\begin{array}{ll} 1 = ab^2 & \\ 1 = 6b^2b^2 = 6b^4 & \text{Substitute } a. \\ b = \left(\dfrac{1}{6}\right)^{\frac{1}{4}} & \text{Use properties of exponents to isolate } b. \\ b \approx 0.6389 & \text{Round 4 decimal places.} \end{array}$$

Use the value of b in the first equation to solve for the value of a :

$$a = 6b^2 \approx 6(0.6389)^2 \approx 2.4492$$

Thus, the equation is $f(x) = 2.4492(0.6389)^x$.

We can graph our model to check our work. Notice that the graph in [link] passes through the initial points given in the problem, $(-2,\ 6)$ and $(2,\ 1)$. The graph is an example of an exponential decay function.



The graph of $f(x) = 2.4492(0.6389)^x$ models exponential decay.

Note:
Exercise:

Problem:

Given the two points $(1, 3)$ and $(2, 4.5)$,find the equation of the exponential function that passes through these two points.

Solution:

$f(x) = 2(1.5)^x$

Note:
Do two points always determine a unique exponential function?
Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in x*, which in many real world cases involves time.*

Note:

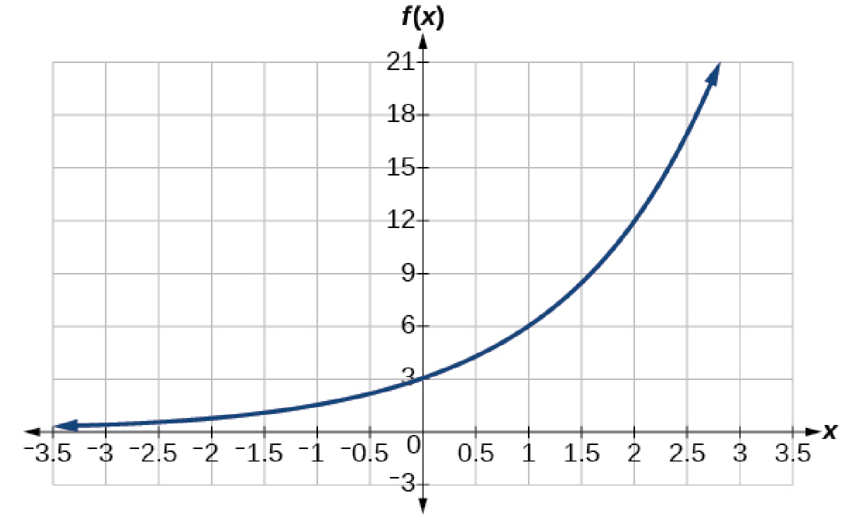
Given the graph of an exponential function, write its equation.

1. First, identify two points on the graph. Choose the *y*-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
2. If one of the data points is the *y*-intercept $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
3. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
4. Write the exponential function, $f(x) = a(b)^x$.

Example:
Exercise:

Problem:
Writing an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in [link].



Solution:

We can choose the *y*-intercept of the graph, $(0, 3)$, as our first point. This gives us the initial value, $a = 3$. Next, choose a point on the curve some distance away from $(0, 3)$ that has integer coordinates. One such point is $(2, 12)$.

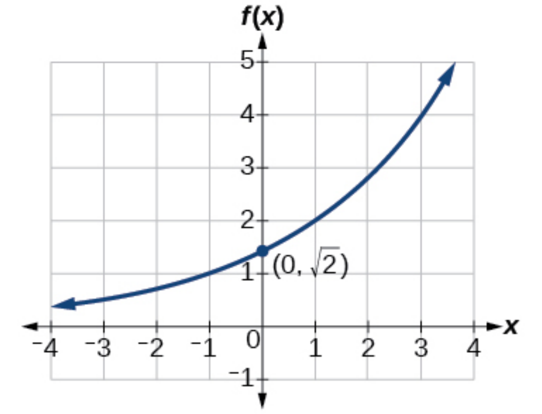
Equation:

$$\begin{array}{ll} y = ab^x & \text{Write the general form of an exponential equation.} \\ y = 3b^x & \text{Substitute the initial value 3 for } a. \\ 12 = 3b^2 & \text{Substitute in 12 for } y \text{ and 2 for } x. \\ 4 = b^2 & \text{Divide by 3.} \\ b = \pm 2 & \text{Take the square root.} \end{array}$$

Because we restrict ourselves to positive values of b, we will use $b = 2$. Substitute a and b into the standard form to yield the equation $f(x) = 3(2)^x$.

Note:
Exercise:

Problem: Find an equation for the exponential function graphed in [link].



Solution:

$f(x) = \sqrt{2}\left(\sqrt{2}\right)^x$. Answers may vary due to round-off error. The answer should be very close to $1.4142(1.4142)^x$.

Note:
Given two points on the curve of an exponential function, use a graphing calculator to find the equation.

1. Press **[STAT].**
2. Clear any existing entries in columns **L1** or **L2.**
3. In **L1**, enter the *x*-coordinates given.
4. In **L2**, enter the corresponding *y*-coordinates.
5. Press **[STAT]** again. Cursor right to **CALC**, scroll down to **ExpReg (Exponential Regression)**, and press **[ENTER].**
6. The screen displays the values of *a* and *b* in the exponential equation $y = a \cdot b^x$.

Example:
Exercise:

Problem:
Using a Graphing Calculator to Find an Exponential Function

Use a graphing calculator to find the exponential equation that includes the points $(2, 24.8)$ and $(5, 198.4)$.

Solution:

Follow the guidelines above. First press **[STAT]**, **[EDIT]**, **[1: Edit…],** and clear the lists **L1** and **L2**. Next, in the **L1** column, enter the *x*-coordinates, 2 and 5. Do the same in the **L2** column for the *y*-coordinates, 24.8 and 198.4.

Now press **[STAT]**, **[CALC]**, **[0: ExpReg]** and press **[ENTER]**. The values $a = 6.2$ and $b = 2$ will be displayed. The exponential equation is $y = 6.2 \cdot 2^x$.

Note:
Exercise:

Problem:

Use a graphing calculator to find the exponential equation that includes the points (3, 75.98) and (6, 481.07).

Solution:

$y \approx 12 \cdot 1.85^x$

Applying the Compound-Interest Formula

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use **compound interest**. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time t, principal P, APR r, and number of compounding periods in a year n :
Equation:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, observe [link], which shows the result of investing $1,000 at 10% for one year. Notice how the value of the account increases as the compounding frequency increases.

Frequency	Value after 1 year
Annually	\$1100
Semiannually	\$1102.50
Quarterly	\$1103.81
Monthly	\$1104.71
Daily	\$1105.16

Note:
The Compound Interest Formula
Compound interest can be calculated using the formula
Equation:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where

- $A(t)$ is the account value,
- t is measured in years,
- P is the starting amount of the account, often called the principal, or more generally present value,
- r is the annual percentage rate (APR) expressed as a decimal, and
- n is the number of compounding periods in one year.

Example:
Exercise:

Problem:
Calculating Compound Interest

If we invest \$3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution:

Because we are starting with \$3,000, $P = 3000$. Our interest rate is 3%, so $r = 0.03$. Because we are compounding quarterly, we are compounding 4 times per year, so $n = 4$. We want to know the value of the account in 10 years, so we are looking for $A(10)$,the value when $t = 10$.
Equation:

$$\begin{aligned} A(t) &= P\left(1 + \tfrac{r}{n}\right)^{nt} && \text{Use the compound interest formula.} \\ A(10) &= 3000\left(1 + \tfrac{0.03}{4}\right)^{4 \cdot 10} && \text{Substitute using given values.} \\ &\approx \$4045.05 && \text{Round to two decimal places.} \end{aligned}$$

The account will be worth about $4,045.05 in 10 years.

Note:
Exercise:

Problem:

An initial investment of $100,000 at 12% interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?

Solution:

about $3,644,675.88

Example:
Exercise:

Problem:
Using the Compound Interest Formula to Solve for the Principal

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $40,000 over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

Solution:

The nominal interest rate is 6%, so $r = 0.06$. Interest is compounded twice a year, so $k = 2$.

We want to find the initial investment, P, needed so that the value of the account will be worth $40,000 in 18 years. Substitute the given values into the compound interest formula, and solve for P.
Equation:

$$\begin{aligned}
A(t) &= P\left(1+\frac{r}{n}\right)^{nt} && \text{Use the compound interest formula.} \\
40{,}000 &= P\left(1+\frac{0.06}{2}\right)^{2(18)} && \text{Substitute using given values } A,\ r,\ n, \text{ and } t. \\
40{,}000 &= P(1.03)^{36} && \text{Simplify.} \\
\frac{40{,}000}{(1.03)^{36}} &= P && \text{Isolate } P. \\
P &\approx \$13,801 && \text{Divide and round to the nearest dollar.}
\end{aligned}$$

Lily will need to invest $13,801 to have $40,000 in 18 years.

Note:
Exercise:

Problem:

Refer to [link]. To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?

Solution:

$13,693

Evaluating Functions with Base *e*

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. [link] shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of $1 invested at 100% interest for 1 year, compounded at various frequencies, listed in [link].

Frequency	$A(t)=\left(1+\frac{1}{n}\right)^n$	**Value**
Annually	$\left(1+\frac{1}{1}\right)^1$	$2
Semiannually	$\left(1+\frac{1}{2}\right)^2$	$2.25
Quarterly	$\left(1+\frac{1}{4}\right)^4$	$2.441406
Monthly	$\left(1+\frac{1}{12}\right)^{12}$	$2.613035
Daily	$\left(1+\frac{1}{365}\right)^{365}$	$2.714567
Hourly	$\left(1+\frac{1}{8760}\right)^{8760}$	$2.718127
Once per minute	$\left(1+\frac{1}{525600}\right)^{525600}$	$2.718279
Once per second	$\left(1+\frac{1}{31536000}\right)^{31536000}$	$2.718282

These values appear to be approaching a limit as n increases without bound. In fact, as n gets larger and larger, the expression $\left(1+\frac{1}{n}\right)^n$ approaches a number used so frequently in mathematics that it has its own name: the letter e. This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

Note:
The Number e
The letter e represents the irrational number
Equation:

$$\left(1+\frac{1}{n}\right)^{n}, \text{as } n \text{ increases without bound}$$

The letter e is used as a base for many real-world exponential models. To work with base e, we use the approximation, $e \approx 2.718282.$ The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

Example:
Exercise:

Problem:
Using a Calculator to Find Powers of *e*

Calculate $e^{3.14}.$ Round to five decimal places.

Solution:

On a calculator, press the button labeled $[e^x].$ The window shows $[e \text{ ^ } (\ ].$ Type 3.14 and then close parenthesis, $[)].$ Press [ENTER]. Rounding to 5 decimal places, $e^{3.14} \approx 23.10387.$ Caution: Many scientific calculators have an “Exp” button, which is used to enter numbers in scientific notation. It is not used to find powers of e.

Note:
Exercise:

Problem: Use a calculator to find $e^{-0.5}.$ Round to five decimal places.

Solution:

$e^{-0.5} \approx 0.60653$

Investigating Continuous Growth

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, e is used as the base for exponential functions. Exponential models that use e as the base are called *continuous growth or decay models*. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

Note:
The Continuous Growth/Decay Formula

For all real numbers t,and all positive numbers a and r,continuous growth or decay is represented by the formula

Equation:

$$A(t) = ae^{rt}$$

where

- a is the initial value,
- r is the continuous growth rate per unit time,
- and t is the elapsed time.

If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

Equation:

$$A(t) = Pe^{rt}$$

where

- P is the principal or the initial invested,
- r is the growth or interest rate per unit time,
- and t is the period or term of the investment.

Note:

Given the initial value, rate of growth or decay, and time t, solve a continuous growth or decay function.

1. Use the information in the problem to determine a, the initial value of the function.
2. Use the information in the problem to determine the growth rate r.
    a. If the problem refers to continuous growth, then $r > 0$.
    b. If the problem refers to continuous decay, then $r < 0$.
3. Use the information in the problem to determine the time t.
4. Substitute the given information into the continuous growth formula and solve for $A(t)$.

Example:

Exercise:

Problem:

Calculating Continuous Growth

A person invested \$1,000 in an account earning a nominal 10% per year compounded continuously. How much was in the account at the end of one year?

Solution:

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10$. The initial investment was $1,000, so $P = 1000$. We use the continuous compounding formula to find the value after $t = 1$ year:

Equation:

$$\begin{aligned} A(t) &= Pe^{rt} && \text{Use the continuous compounding formula.} \\ &= 1000(e)^{0.1} && \text{Substitute known values for } P,\ r,\ \text{and } t. \\ &\approx 1105.17 && \text{Use a calculator to approximate.} \end{aligned}$$

The account is worth $1,105.17 after one year.

Note:

Exercise:

Problem:

A person invests $100,000 at a nominal 12% interest per year compounded continuously. What will be the value of the investment in 30 years?

Solution:

$3,659,823.44

Example:

Exercise:

Problem:

Calculating Continuous Decay

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

Solution:

Since the substance is decaying, the rate, 17.3%, is negative. So, $r = -0.173$. The initial amount of radon-222 was 100 mg, so $a = 100$. We use the continuous decay formula to find the value after $t = 3$ days:

Equation:

$$\begin{aligned} A(t) &= ae^{rt} && \text{Use the continuous growth formula.} \\ &= 100e^{-0.173(3)} && \text{Substitute known values for } a,\ r,\ \text{and } t. \\ &\approx 59.5115 && \text{Use a calculator to approximate.} \end{aligned}$$

So 59.5115 mg of radon-222 will remain.

Note:
Exercise:

Problem: Using the data in [link], how much radon-222 will remain after one year?

Solution:

3.77E-26 (This is calculator notation for the number written as 3.77×10^{-26} in scientific notation. While the output of an exponential function is never zero, this number is so close to zero that for all practical purposes we can accept zero as the answer.)

Note:
Access these online resources for additional instruction and practice with exponential functions.

- Exponential Growth Function
- Compound Interest

Key Equations

definition of the exponential function	$f(x) = b^x, \text{ where } b > 0, \ b \neq 1$
definition of exponential growth	$f(x) = ab^x, \text{ where } a > 0, b > 0, b \neq 1$
compound interest formula	$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}, \text{ where}$ $A(t)$ is the account value at time t t is the number of years P is the initial investment, often called the principal r is the annual percentage rate (APR), or nominal rate n is the number of compounding periods in one year
continuous growth formula	$A(t) = ae^{rt}, \text{ where}$ $t$is the number of unit time periods of growth $a$is the starting amount (in the continuous compounding formula a is replaced with P, the principal) $e$is the mathematical constant, $e \approx 2.718282$

Key Concepts

- An exponential function is defined as a function with a positive constant other than 1 raised to a variable exponent. See [link].
- A function is evaluated by solving at a specific value. See [link] and [link].
- An exponential model can be found when the growth rate and initial value are known. See [link].
- An exponential model can be found when the two data points from the model are known. See [link].
- An exponential model can be found using two data points from the graph of the model. See [link].
- An exponential model can be found using two data points from the graph and a calculator. See [link].
- The value of an account at any time t can be calculated using the compound interest formula when the principal, annual interest rate, and compounding periods are known. See [link].
- The initial investment of an account can be found using the compound interest formula when the value of the account, annual interest rate, compounding periods, and life span of the account are known. See [link].
- The number e is a mathematical constant often used as the base of real world exponential growth and decay models. Its decimal approximation is $e \approx 2.718282$.
- Scientific and graphing calculators have the key $[e^x]$ or $[\exp(x)]$ for calculating powers of e. See [link].
- Continuous growth or decay models are exponential models that use e as the base. Continuous growth and decay models can be found when the initial value and growth or decay rate are known. See [link] and [link].

Section Exercises

Verbal

Exercise:

Problem:

Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.

Solution:

Linear functions have a constant rate of change. Exponential functions increase based on a percent of the original.

Exercise:

Problem:

Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.

Exercise:

Problem:

The Oxford Dictionary defines the word *nominal* as a value that is “stated or expressed but not necessarily corresponding exactly to the real value.”[footnote] Develop a reasonable argument for why the term *nominal rate* is used to describe the annual percentage rate of an investment account that compounds interest.
Oxford Dictionary. http://oxforddictionaries.com/us/definition/american_english/nomina.

Solution:

When interest is compounded, the percentage of interest earned to principal ends up being greater than the annual percentage rate for the investment account. Thus, the annual percentage rate does not necessarily correspond to the real interest earned, which is the very definition of *nominal*.

Algebraic

For the following exercises, identify whether the statement represents an exponential function. Explain.

Exercise:

Problem: The average annual population increase of a pack of wolves is 25.

Exercise:

Problem: A population of bacteria decreases by a factor of $\frac{1}{8}$ every 24 hours.

Solution:

exponential; the population decreases by a proportional rate. .

Exercise:

Problem: The value of a coin collection has increased by 3.25% annually over the last 20 years.

Exercise:

Problem:

For each training session, a personal trainer charges his clients $\$5$ less than the previous training session.

Solution:

not exponential; the charge decreases by a constant amount each visit, so the statement represents a linear function. .

Exercise:

Problem: The height of a projectile at time t is represented by the function $h(t) = -4.9t^2 + 18t + 40$.

For the following exercises, consider this scenario: For each year t,the population of a forest of trees is represented by the function $A(t) = 115(1.025)^t$. In a neighboring forest, the population of the same type of tree is represented by the function $B(t) = 82(1.029)^t$. (Round answers to the nearest whole number.)

Exercise:

Problem: Which forest’s population is growing at a faster rate?

Solution:

The forest represented by the function $B(t) = 82(1.029)^t$.

Exercise:

Problem: Which forest had a greater number of trees initially? By how many?

Exercise:

Problem:

Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 20 years? By how many?

Solution:

After $t = 20$ years, forest A will have 43 more trees than forest B.

Exercise:

Problem:

Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 100 years? By how many?

Exercise:

Problem:

Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

Solution:

Answers will vary. Sample response: For a number of years, the population of forest A will increasingly exceed forest B, but because forest B actually grows at a faster rate, the population will eventually become larger than forest A and will remain that way as long as the population growth models hold. Some factors that might influence the long-term validity of the exponential growth model are drought, an epidemic that culls the population, and other environmental and biological factors.

For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain.

Exercise:

Problem: $y = 300(1-t)^5$

Exercise:

Problem: $y = 220(1.06)^x$

Solution:

exponential growth; The growth factor, 1.06, is greater than 1.

Exercise:

Problem: $y = 16.5(1.025)^{\frac{1}{x}}$

Exercise:

Problem: $y = 11,701(0.97)^t$

Solution:

exponential decay; The decay factor, 0.97, is between 0 and 1.

For the following exercises, find the formula for an exponential function that passes through the two points given.

Exercise:

Problem: $(0, 6)$ and $(3, 750)$

Exercise:

Problem: $(0, 2000)$ and $(2, 20)$

Solution:

$f(x) = 2000(0.1)^x$

Exercise:

Problem: $\left(-1, \frac{3}{2}\right)$ and $(3, 24)$

Exercise:

Problem: $(-2, 6)$ and $(3, 1)$

Solution:

$f(x) = \left(\frac{1}{6}\right)^{-\frac{3}{5}}\left(\frac{1}{6}\right)^{\frac{x}{5}} \approx 2.93(0.699)^x$

Exercise:

Problem: $(3, 1)$ and $(5, 4)$

For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

Exercise:

Problem:

x	1	2	3	4
$f(x)$	70	40	10	-20

Solution:

Linear

Exercise:

Problem:

x	1	2	3	4
$h(x)$	70	49	34.3	24.01

Exercise:

Problem:

x	1	2	3	4
$m(x)$	80	61	42.9	25.61

Solution:

Neither

Exercise:

Problem:

x	1	2	3	4
$f(x)$	10	20	40	80

Exercise:

Problem:

x	1	2	3	4
$g(x)$	-3.25	2	7.25	12.5

Solution:

Linear

For the following exercises, use the compound interest formula, $A(t)=P\left(1+\frac{r}{n}\right)^{nt}$.

Exercise:

Problem:

After a certain number of years, the value of an investment account is represented by the equation $10,250\left(1+\frac{0.04}{12}\right)^{120}$. What is the value of the account?

Exercise:

Problem: What was the initial deposit made to the account in the previous exercise?

Solution:

$\$10,250$

Exercise:

Problem: How many years had the account from the previous exercise been accumulating interest?

Exercise:

Problem:

An account is opened with an initial deposit of $6,500 and earns 3.6% interest compounded semi-annually. What will the account be worth in 20 years?

Solution:

$\$13,268.58$

Exercise:

Problem:

How much more would the account in the previous exercise have been worth if the interest were compounding weekly?

Exercise:

Problem: Solve the compound interest formula for the principal, P.

Solution:

$P=A(t)\cdot\left(1+\frac{r}{n}\right)^{-nt}$

Exercise:

Problem:

Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $\$14,472.74$ after earning 5.5% interest compounded monthly for 5 years. (Round to the nearest dollar.)

Exercise:

Problem:

How much more would the account in the previous two exercises be worth if it were earning interest for 5 more years?

Solution:

$\$4{,}572.56$

Exercise:

Problem:

Use properties of rational exponents to solve the compound interest formula for the interest rate, r.

Exercise:

Problem:

Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of $9,000 and was worth $13,373.53 after 10 years.

Solution:

4%

Exercise:

Problem:

Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of $5,500, and was worth $38,455 after 30 years.

For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain.

Exercise:

Problem: $y = 3742(e)^{0.75t}$

Solution:

continuous growth; the growth rate is greater than 0.

Exercise:

Problem: $y = 150(e)^{\frac{3.25}{t}}$

Exercise:

Problem: $y = 2.25(e)^{-2t}$

Solution:

continuous decay; the growth rate is less than 0.

Exercise:

Problem:

Suppose an investment account is opened with an initial deposit of $\$12,000$ earning 7.2% interest compounded continuously. How much will the account be worth after 30 years?

Exercise:

Problem:

How much less would the account from Exercise 42 be worth after 30 years if it were compounded monthly instead?

Solution:

$\$669.42$

Numeric

For the following exercises, evaluate each function. Round answers to four decimal places, if necessary.

Exercise:

Problem: $f(x) = 2(5)^x$, for $f\,(-3)$

Exercise:

Problem: $f(x) = -4^{2x+3}$, for $f\,(-1)$

Solution:

$f(-1) = -4$

Exercise:

Problem: $f(x) = e^x$, for $f\,(3)$

Exercise:

Problem: $f(x) = -2e^{x-1}$, for $f\,(-1)$

Solution:

$f(-1) \approx -0.2707$

Exercise:

Problem: $f(x)=2.7(4)^{-x+1}+1.5$, for$f\left(-2\right)$

Exercise:

Problem: $f(x)=1.2e^{2x}-0.3$, for $f\left(3\right)$

Solution:

$f(3)\approx 483.8146$

Exercise:

Problem: $f(x)=-\frac{3}{2}(3)^{-x}+\frac{3}{2}$, for $f\left(2\right)$

Technology

For the following exercises, use a graphing calculator to find the equation of an exponential function given the points on the curve.

Exercise:

Problem: $(0,3)$ and $(3,375)$

Solution:

$y=3\cdot 5^{x}$

Exercise:

Problem: $(3,222.62)$ and $(10,77.456)$

Exercise:

Problem: $(20,29.495)$ and $(150,730.89)$

Solution:

$y\approx 18\cdot 1.025^{x}$

Exercise:

Problem: $(5,2.909)$ and $(13,0.005)$

Exercise:

Problem: $\left(11\text{,}310.035\right)$ and $\left(25\text{,}356.3652\right)$

Solution:

$y\approx 0.2\cdot 1.95^{x}$

Extensions

Exercise:

Problem:

The *annual percentage yield* (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $\text{APY} = \left(1+\frac{r}{12}\right)^{12} - 1$.

Exercise:

Problem:

Repeat the previous exercise to find the formula for the APY of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I(n)$ for the APY of any account that compounds n times per year.

Solution:

$$\text{APY} = \frac{A(t)-a}{a} = \frac{a\left(1+\frac{r}{365}\right)^{365(1)}-a}{a} = \frac{a\left[\left(1+\frac{r}{365}\right)^{365}-1\right]}{a} = \left(1+\frac{r}{365}\right)^{365}-1; I(n) = \left(1+\frac{r}{n}\right)^{n}-1$$

Exercise:

Problem:

Recall that an exponential function is any equation written in the form $f(x) = a \cdot b^x$ such that a and b are positive numbers and $b \neq 1$. Any positive number b can be written as $b = e^n$ for some value of n. Use this fact to rewrite the formula for an exponential function that uses the number e as a base.

Exercise:

Problem:

In an exponential decay function, the base of the exponent is a value between 0 and 1. Thus, for some number $b > 1$, the exponential decay function can be written as $f(x) = a \cdot \left(\frac{1}{b}\right)^x$. Use this formula, along with the fact that $b = e^n$, to show that an exponential decay function takes the form $f(x) = a(e)^{-nx}$ for some positive number n .

Solution:

Let f be the exponential decay function $f(x) = a \cdot \left(\frac{1}{b}\right)^x$ such that $b > 1$. Then for some number $n > 0$, $f(x) = a \cdot \left(\frac{1}{b}\right)^x = a\left(b^{-1}\right)^x = a\left(\left(e^n\right)^{-1}\right)^x = a\left(e^{-n}\right)^x = a(e)^{-nx}$.

Exercise:

Problem:

The formula for the amount A in an investment account with a nominal interest rate r at any time t is given by $A(t) = a(e)^{rt}$,where a is the amount of principal initially deposited into an account that compounds continuously. Prove that the percentage of interest earned to principal at any time t can be calculated with the formula $I(t) = e^{rt} - 1$.

Real-World Applications

Exercise:

Problem:

The fox population in a certain region has an annual growth rate of 9% per year. In the year 2012, there were 23,900 fox counted in the area. What is the fox population predicted to be in the year 2020?

Solution:

$47,622\text{ fox}$

Exercise:

Problem:

A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50mg of the substance remains. How many milligrams will remain after 54 hours?

Exercise:

Problem:

In the year 1985, a house was valued at $110,000. By the year 2005, the value had appreciated to $145,000. What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?

Solution:

$1.39\%;\ \$155,368.09$

Exercise:

Problem:

A car was valued at $38,000 in the year 2007. By 2013, the value had depreciated to $11,000 If the car's value continues to drop by the same percentage, what will it be worth by 2017?

Exercise:

Problem:

Jamal wants to save $54,000 for a down payment on a home. How much will he need to invest in an account with 8.2% APR, compounding daily, in order to reach his goal in 5 years?

Solution:

$\$35,838.76$

Exercise:

Problem:

Kyoko has $10,000 that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $15,000 by the time she finishes graduate school in 6 years. To the nearest hundredth of a percent, what should her minimum annual interest rate be in order to reach her goal? (*Hint*: solve the compound interest formula for the interest rate.)

Exercise:

Problem:

Alyssa opened a retirement account with 7.25% APR in the year 2000. Her initial deposit was $13,500. How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?

Solution:

$\$82,247.78;\ \$449.75$

Exercise:

Problem:

An investment account with an annual interest rate of 7% was opened with an initial deposit of $4,000 Compare the values of the account after 9 years when the interest is compounded annually, quarterly, monthly, and continuously.

Glossary

annual percentage rate (APR)	the yearly interest rate earned by an investment account, also called *nominal rate*
compound interest	interest earned on the total balance, not just the principal
exponential growth	a model that grows by a rate proportional to the amount present
nominal rate	the yearly interest rate earned by an investment account, also called *annual percentage rate*

Exponential Function - Resources
Exponential function resources around the web

Learning Activity: Exponential Functions

Watch Video: Evaluating an exponential function.

Khan Academy Lesson: Introduction to Exponential Functions.

Watch Video: Solving exponential equations graphically.

Using Interval Notation

Interval notation is another method for writing domain and range.

In set builder notation braces (curly parentheses {}) and variables are used to express the domain and range. Interval notation is often considered more efficient.

In interval notation, there are only 5 symbols to know:

- Open parentheses ( )
- Closed parentheses [ ]
- Infinity ∞
- Negative Infinity $-\infty$
- Union Sign $\cup$

To use interval notation:

Use the open parentheses ( ) if the value is not included in the graph. (i.e. the graph is undefined at that point... there's a hole or asymptote, or a jump)

If the graph goes on forever to the left, the domain will start with ($-\infty$. If the graph travels downward forever, the range will start with ($-\infty$. Similarly, if the graph goes on forever at the right or up, end with ∞)

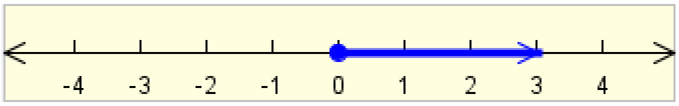
Use the brackets [ ] if the value is part of the graph.

Whenever there is a break in the graph, write the interval up to the point. Then write another interval for the section of the graph after that part. Put a union sign between each interval to "join" them together.

Now for some practice so you can see if any of this makes sense.

Write the following using interval notation:
Exercise:

Problem:

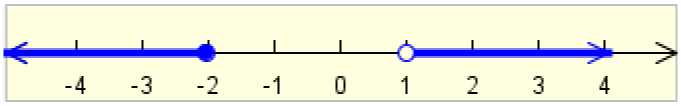


Solution:

$[0, \infty)$

Exercise:

Problem:

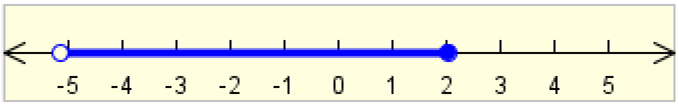


Solution:

$(-\infty, -2] \cup (1, \infty)$

Exercise:

Problem:

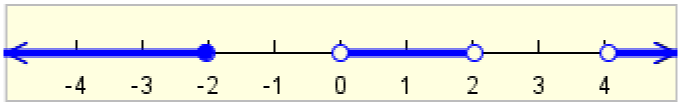

-5
-4
-3
-2
-1
0
1
2
3
4
5

Solution:

$(-5, 2]$

Exercise:

Problem:

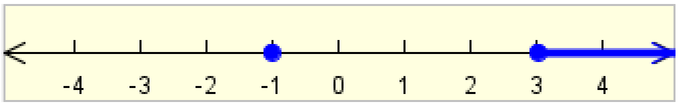


Solution:

$(-\infty, -2] \cup (0, 2) \cup (4, \infty)$

Exercise:

Problem:

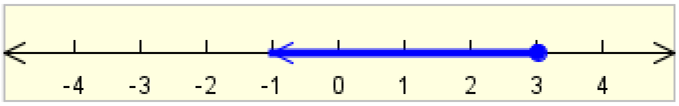


Solution:

$[-1] \cup [3, \infty)$

Exercise:

Problem:



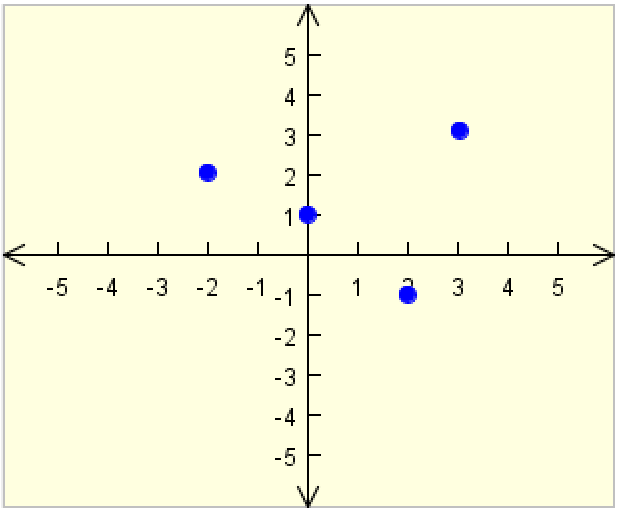
Solution:

$(-\infty, 3]$

Write the domain and range of the following in interval notation:

Exercise:

Problem:



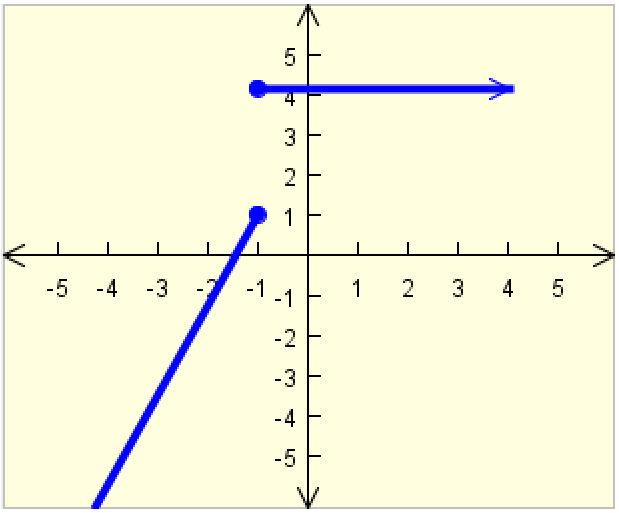
Solution:

Domain: $[-2] \cup [0] \cup [2] \cup [3]$

Range: $[-1] \cup [1] \cup [2] \cup [3]$

Exercise:

Problem:



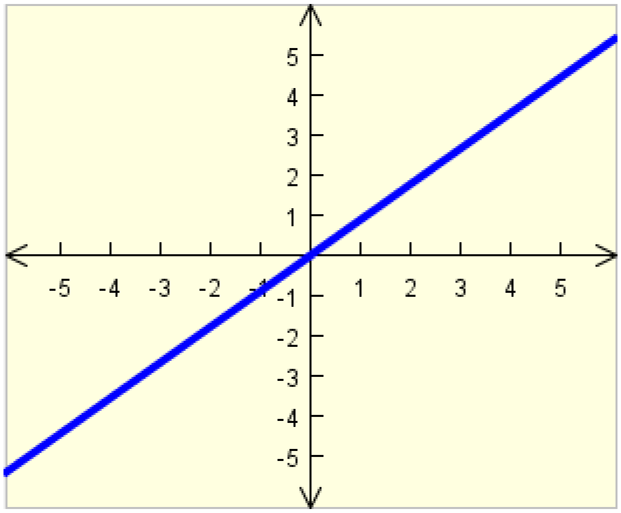
Solution:

Domain: $(-\infty, \infty)$

Range: $(\infty, 1] \cup [4]$

Exercise:

Problem:



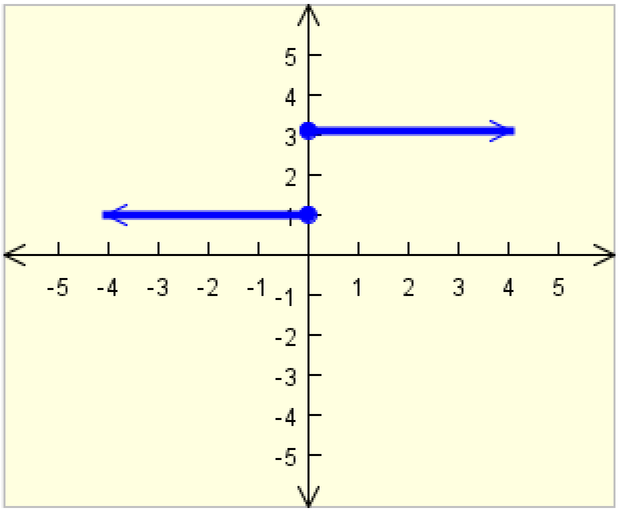
Solution:

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Exercise:

Problem:



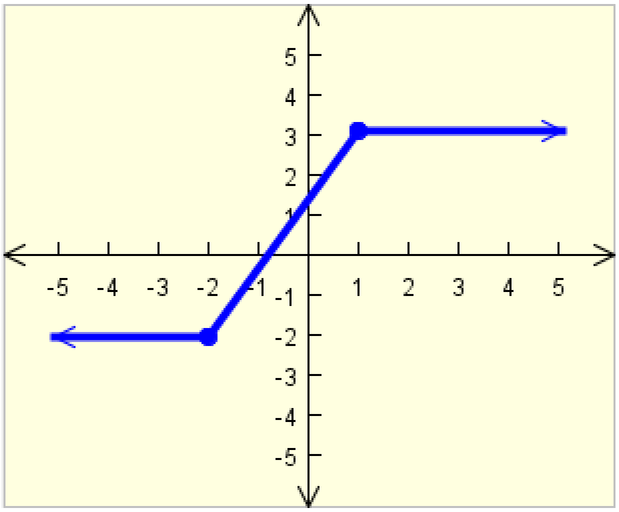
Solution:

Domain: $(-\infty, \infty)$

Range: $[1] \cup [3]$

Exercise:

Problem:



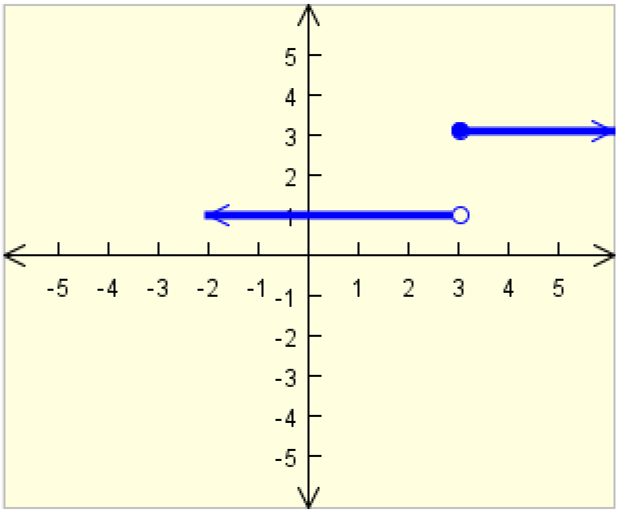
Solution:

Domain: $(-\infty, \infty)$

Range: $[-2, 3]$

Exercise:

Problem:



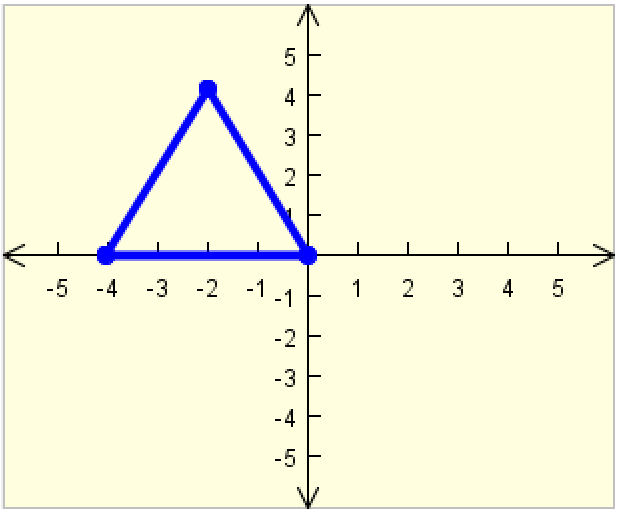
Solution:

Domain: $(-\infty, \infty)$

Range: $[1] \cup [3]$

Exercise:

Problem:



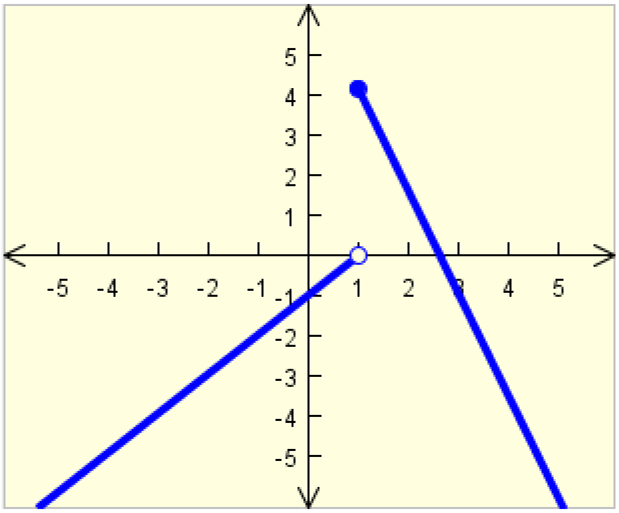
Solution:

Domain: $[-4, 0]$

Range: $[0, 4]$

Exercise:

Problem:



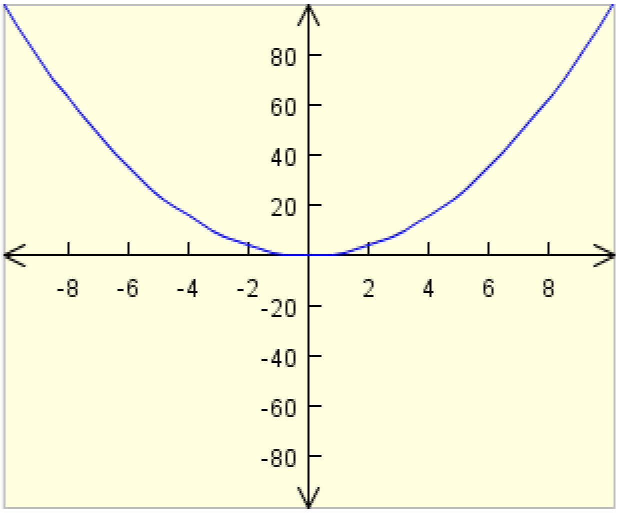
Solution:

Domain: $(-\infty, \infty)$

Range: $(-\infty, 4]$

Exercise:

Problem:



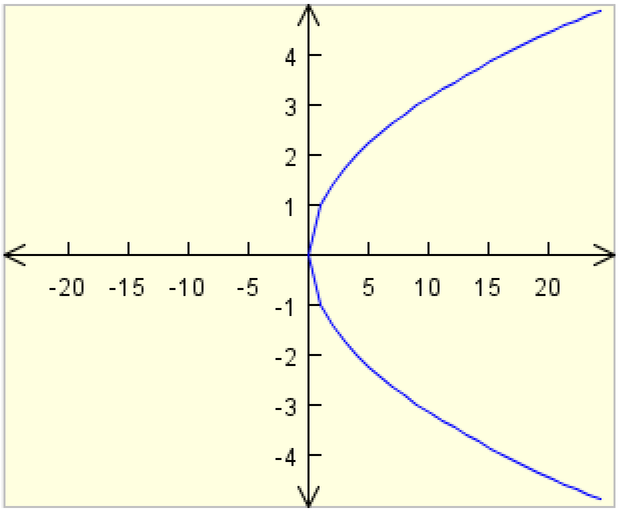
Solution:

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Exercise:

Problem:



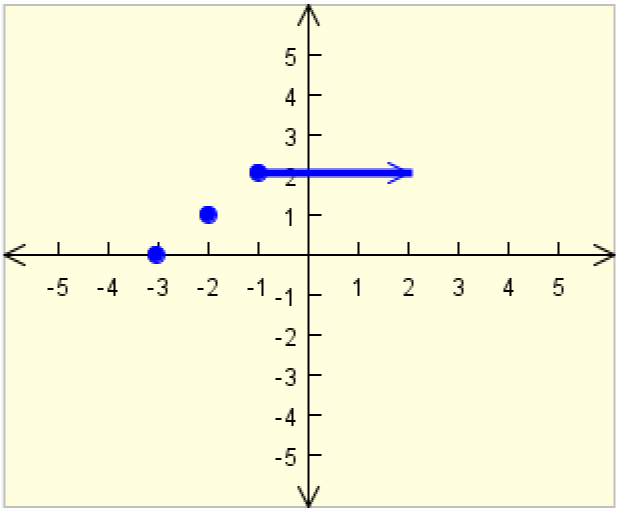
Solution:

Domain: $[0, \infty)$

Range: $(-\infty, \infty)$

Exercise:

Problem:



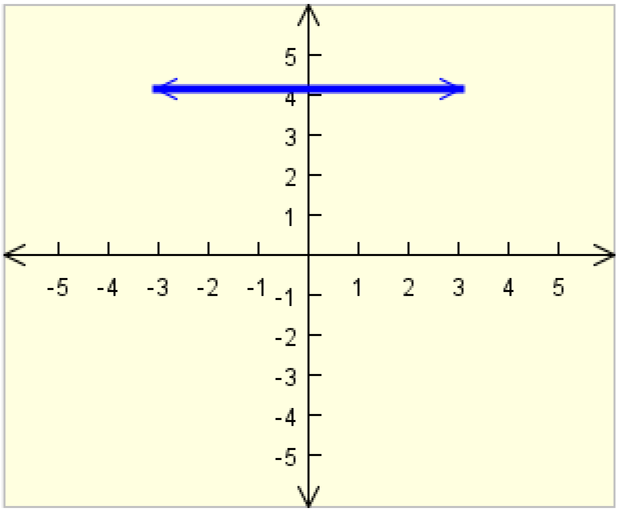
Solution:

Domain: $[-3] \cup [-2] \cup [-1, \infty)$

Range: $[0] \cup [1] \cup [2]$

Exercise:

Problem:



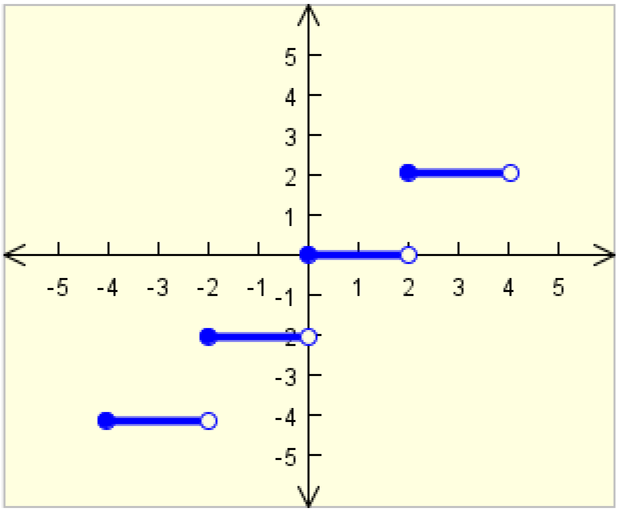
Solution:

Domain: $(-\infty, \infty)$

Range: $[4]$

Exercise:

Problem:



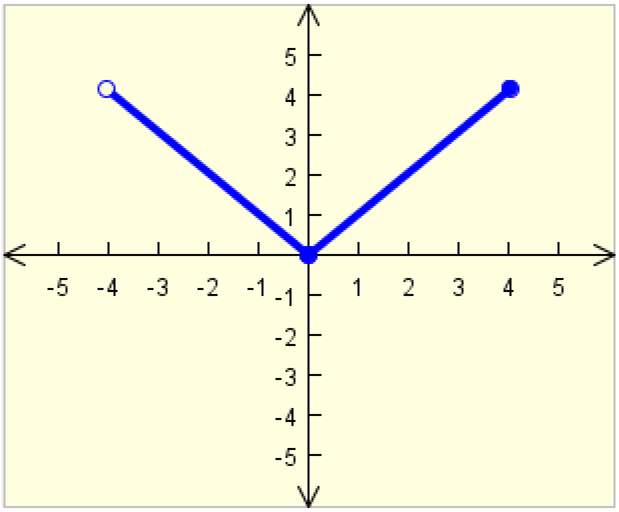
Solution:

Domain: $[-4, 4)$

Range: $[-4] \cup [-2] \cup [0] \cup [2]$

Exercise:

Problem:



Solution:

Domain: $(-4, 4]$

Range: $[0, 4]$

Translate Mathematical Phrases
Translate Mathematical Phrases

Translate Mathematical Phrases

Practice with Phrases involving Inquality @onemathematicalcat.org

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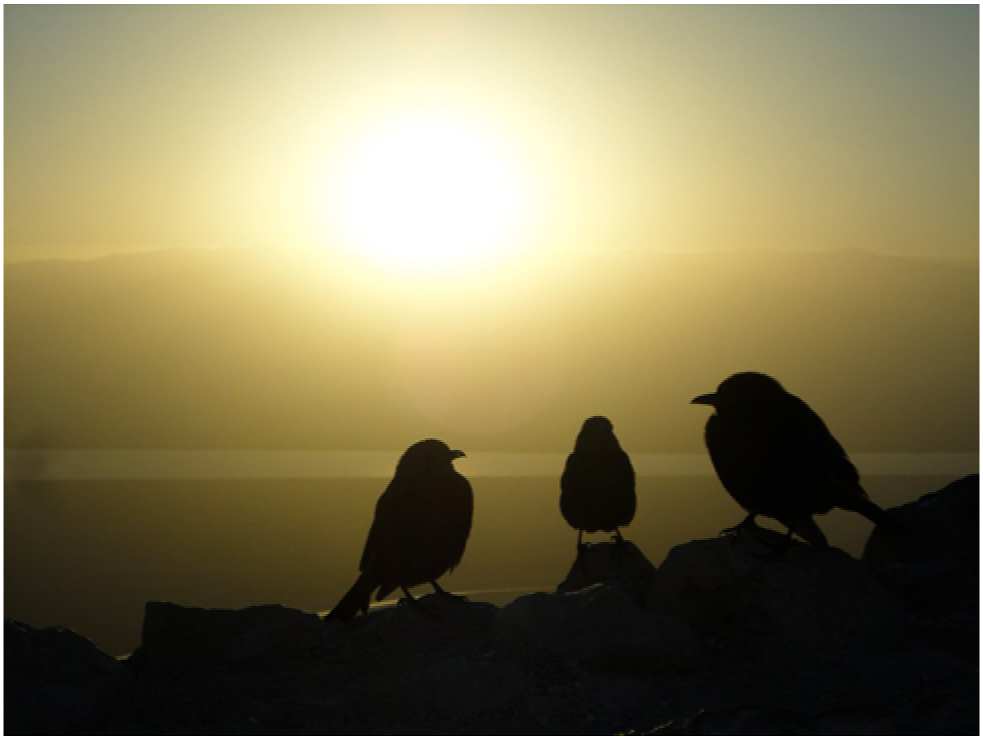
Power Functions

In this section, you will:

- Identify power functions.
- Identify end behavior of power functions.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.



(credit: Jason Bay, Flickr)

Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown in [link].

Year					

	2009	2010	2011	2012	2013
Bird Population	800	897	992	1,083	1,169

The population can be estimated using the function $P(t) = -0.3t^3 + 97t + 800,$ where $P(t)$ represents the bird population on the island t years after 2009. We can use this model to estimate the maximum bird population and when it will occur. We can also use this model to predict when the bird population will disappear from the island. In this section, we will examine functions that we can use to estimate and predict these types of changes.

Identifying Power Functions

Before we can understand the bird problem, it will be helpful to understand a different type of function. A **power function** is a function with a single term that is the product of a real number, a **coefficient,** and a variable raised to a fixed real number.

As an example, consider functions for area or volume. The function for the area of a circle with radius r is

Equation:

$$A(r) = \pi r^2$$

and the function for the volume of a sphere with radius r is

Equation:

$$V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi,$ multiplied by a variable r raised to a power.

Note:
Power Function
A **power function** is a function that can be represented in the form
Equation:

$$f(x) = kx^p$$

where k and p are real numbers, and k is known as the **coefficient**.

Note:
Is $f(x) = 2^x$ a power function?
No. A power function contains a variable base raised to a fixed power. This function has a constant base raised to a variable power. This is called an exponential function, not a power function.

Example:
Exercise:

Problem:
Identifying Power Functions

Which of the following functions are power functions?

$f(x) = 1$	Constant function
$f(x) = x$	Identify function
$f(x) = x^2$	Quadratic function
$f(x) = x^3$	Cubic function
$f(x) = \frac{1}{x}$	Reciprocal function
$f(x) = \frac{1}{x^2}$	Reciprocal squared function
$f(x) = \sqrt{x}$	Square root function
$f(x) = \sqrt[3]{x}$	Cube root function

Solution:

All of the listed functions are power functions.

The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.

The reciprocal and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and cube root functions are power functions with fractional powers because they can be written as $f(x) = x^{\frac{1}{2}}$ or $f(x) = x^{\frac{1}{3}}$.

Note:
Exercise:

Problem: Which functions are power functions?

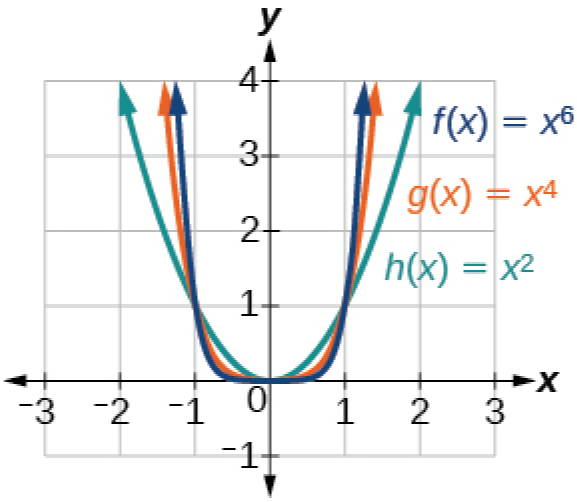
$$\begin{aligned} f(x) &= 2x \cdot 4x^3 \\ g(x) &= -x^5 + 5x^3 \\ h(x) &= \frac{2x^5 - 1}{3x^2 + 4} \end{aligned}$$

Solution:

$f(x)$ is a power function because it can be written as $f(x) = 8x^5$. The other functions are not power functions.

Identifying End Behavior of Power Functions

[link] shows the graphs of $f(x)=x^2$, $g(x)=x^4$ and $h(x)=x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.



Even-power functions

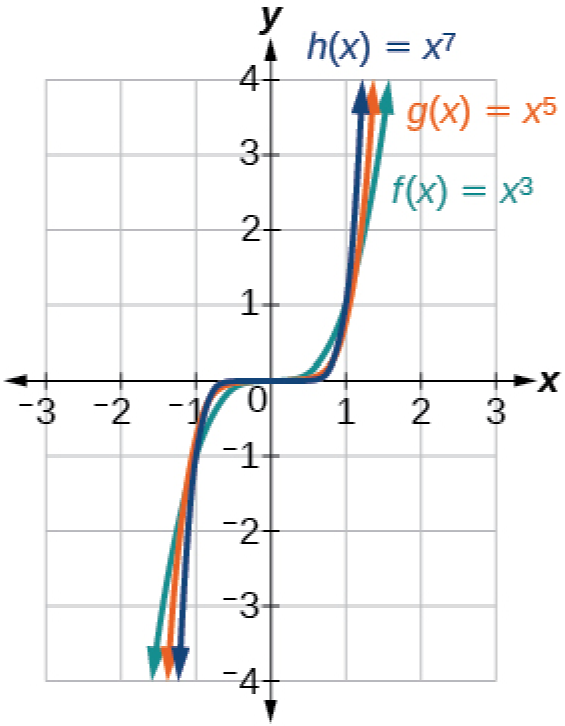
To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that “x approaches infinity,” which can be symbolically written as $x\to\infty,$ we are describing a behavior; we are saying that x is increasing without bound.

With the positive even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f\left(x\right)$ values increase without bound. In symbolic form, we could write

Equation:

$$\text{as } x \to \pm\infty, \ f(x) \to \infty$$

[link] shows the graphs of $f(x) = x^3$, $g(x) = x^5$,and $h(x) = x^7$,which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.



Odd-power functions

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in [link] we see that even functions of the form $f(x) = x^n$, n even, are symmetric about the y-axis. In [link] we see that odd functions of the form $f\left(x\right)=x^n$, n odd, are symmetric about the origin.

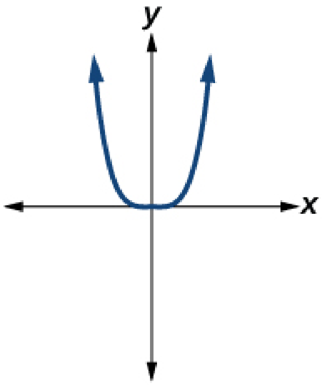
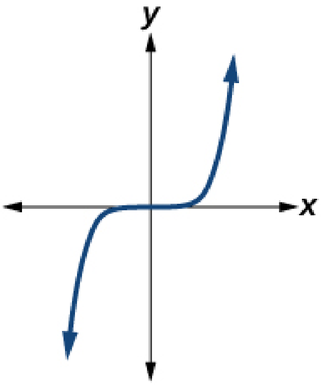
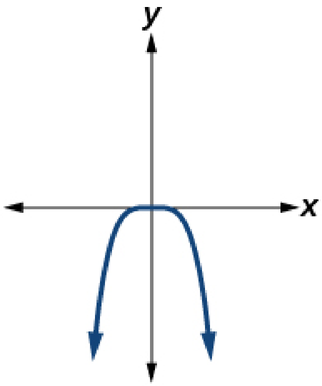
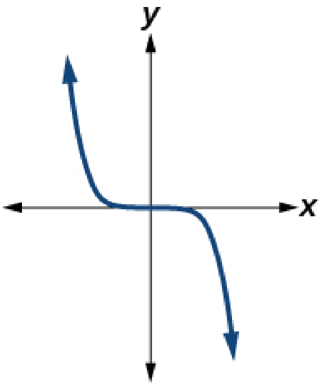
For these odd power functions, as x approaches negative infinity, $f\left(x\right)$ decreases without bound. As x approaches positive infinity, $f\left(x\right)$ increases without bound. In symbolic form we write

Equation:

$$\begin{array}{l}\text{as } x\to -\infty ,\ f(x)\to -\infty \\ \text{as } x\to \infty ,\ f(x)\to \infty \end{array}$$

The behavior of the graph of a function as the input values get very small ($x\to -\infty$) and get very large ($x\to \infty$) is referred to as the **end behavior** of the function. We can use words or symbols to describe end behavior.

[link] shows the end behavior of power functions in the form $f(x)=kx^n$ where n is a non-negative integer depending on the power and the constant.

	Even power	Odd power
Positive constant $k > 0$	 $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to \infty$	 $x \to -\infty, f(x) \to -\infty$ and $x \to \infty, f(x) \to \infty$
Negative constant $k < 0$	 $x \to -\infty, f(x) \to -\infty$ and $x \to \infty, f(x) \to -\infty$	 $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to -\infty$

Note:

Given a power function $f(x) = kx^n$ where n is a non-negative integer, identify the end behavior.

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use [link] to identify the end behavior.

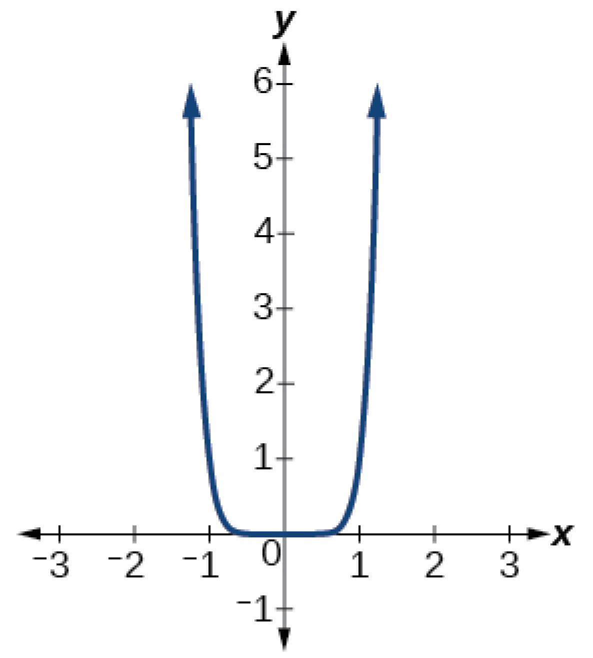
Example:
Exercise:

Problem:
Identifying the End Behavior of a Power Function

Describe the end behavior of the graph of $f(x) = x^8$.

Solution:

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As x approaches infinity, the output (value of $f(x)$) increases without bound. We write as $x \to \infty, f(x) \to \infty.$ As x approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty,\ f(x) \to \infty$. We can graphically represent the function as shown in [link].



Example:
Exercise:

Problem:

Identifying the End Behavior of a Power Function.

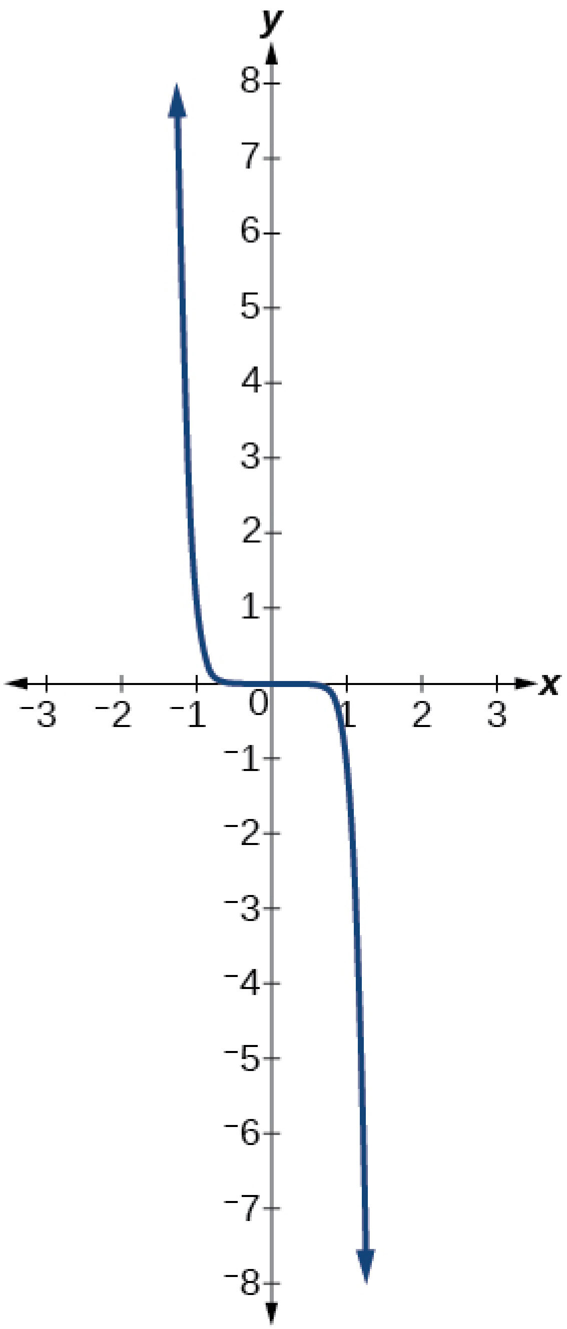
Describe the end behavior of the graph of $f(x) = -x^9$.

Solution:

The exponent of the power function is 9 (an odd number). Because the coefficient is –1 (negative), the graph is the reflection about the x-axis of the graph of $f(x) = x^9$. [link] shows that as x approaches infinity, the output decreases without bound. As x approaches negative infinity, the output increases without bound. In symbolic form, we would write

Equation:

$$\begin{array}{c}\text{as } x \to -\infty, \ f(x) \to \infty \\ \text{as } x \to \infty, \ f(x) \to -\infty\end{array}$$



Analysis

We can check our work by using the table feature on a graphing utility.

x	$f\left(x\right)$
–10	1,000,000,000
–5	1,953,125
0	0
5	–1,953,125
10	–1,000,000,000

We can see from [link] that, when we substitute very small values for x, the output is very large, and when we substitute very large values for x, the output is very small (meaning that it is a very large negative value).

Note:
Exercise:

Problem:

Describe in words and symbols the end behavior of $f(x)=-5x^4$.

Solution:

As x approaches positive or negative infinity, $f\left(x\right)$ decreases without bound: as $x\rightarrow\pm\infty,\ f(x)\rightarrow-\infty$ because of the negative coefficient.

Key Concepts

- A power function is a variable base raised to a number power. See [link].

- The behavior of a graph as the input decreases beyond bound and increases beyond bound is called the end behavior.
- The end behavior depends on whether the power is even or odd. See [link] and [link].

Section Exercises

Verbal

Exercise:

Problem:

Explain the difference between the coefficient of a power function and its degree.

Solution:

The coefficient of the power function is the real number that is multiplied by the variable raised to a power. The degree is the highest power appearing in the function.

Exercise:

Problem:

In general, explain the end behavior of a power function with odd degree if the leading coefficient is positive.

Solution:

As x decreases without bound, so does $f\left(x\right)$. As x increases without bound, so does $f\left(x\right)$.

Algebraic

For the following exercises, identify the function as a power function, a polynomial function, or neither.
Exercise:

Problem: $f(x) = x^5$

Exercise:

Problem: $f(x) = \left(x^2\right)^3$

Solution:

Power function

Exercise:

Problem: $f(x) = x - x^4$

Exercise:

Problem: $f(x) = \frac{x^2}{x^2-1}$

Solution:

Neither

Exercise:

Problem: $f(x) = 2x\,(x+2)(x-1)^2$

Exercise:

Problem: $f(x) = 3^{x+1}$

Solution:

Neither

For the following exercises, determine the end behavior of the functions.

Exercise:

Problem: $f\left(x\right)=x^4$

Solution:

As $x\rightarrow\infty,\ f(x)\rightarrow\infty$, as $x\rightarrow-\infty,\ f(x)\rightarrow\infty$

Exercise:

Problem: $f\left(x\right)=x^3$

Exercise:

Problem: $f\left(x\right)=-x^4$

Solution:

As $x\rightarrow-\infty,\ f(x)\rightarrow-\infty$, as $x\rightarrow\infty,\ f(x)\rightarrow-\infty$

Exercise:

Problem: $f\left(x\right)=-x^9$

Numeric

For the following exercises, make a table to confirm the end behavior of the function.

Exercise:

Problem: $f(x)=-x^3$